Neculai STANCIU

ARTICLES AND NOTES OF MATHEMATICS GYMNASIUM & HIGH SCHOOL

Selections from the book:
Articole şi note de matematică pentru gimnaziu şi liceu

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Dedic această carte soției mele Roxana Mihaela Stanciu şi copiilor noştriBogdan Andrei şi Maria.

Referenți ştiințifici:
Prof. gr. I Constantin Apostol, Colegiul Național „Alexandru Vlahuţã”, Râmnicu Sãrat
Prof. gr. I Gheorghe Ghiţã, Colegiul Național „Mihai Eminescu”, Buzãu
Redactor: Roxana Mihaela Stanciu
Tehnoredactare computerizatã: Roxana Mihaela Stanciu

XIII. Solving problems of concurrence and collinearity using properties of pencils of lines

"Projective geometry is whole geometry"
Arthur Cayley

Neculai N. Stanciu

Abstract. This article is devoted to the study of two fundamental and reciprocal questions: when do three given points lie on a single line, and when do three given lines pass through a single point? The techniques we describe in this article will be augmented by more sophisticated approaches, such as the Papus's theorems, the Desargues's theorems, the Pascal's theorem and the Brianchon's theorem.

The formalism of projective geometry makes a discussion of such properties possible, and exposes some remarkable facts, such as the duality of points and lines.While technique "cross-ratio" of four points, and in the light of duality the cross-ratio of four lines can be useful on contest problems, much of the material here is considered "too advanced" for primary and secondary school education.This is a pity, as some of the most beautiful classical geometry appears only in the projective geometry.
Key words: cross-ratio, bivalent range, harmonic range, harmonic conjugate, concurrence and collinearity .
AMS Classification. 51-xx,51Axx, 51A05.

1**. Main purpose** - of the results below is familiarizing readers with new methods *(little known even teachers of mathematics)* solving problems of concurrence and collinearity namely the techniques offered by pencils of lines properties.
We consider fig.1 where $S(a,b,c,d)$ or $S(A,B,C,D)$ represents a *convergent pencil of lines*, with its own point S and rays a,b,c,d or SA, SB, SC, SD and fig.2 where $S(a,b,c,d)$ is a *parallel pencil of lines* with rays a,b,c,d or SA, SB, SC, SD (S is *improperly point*).

Fig.2

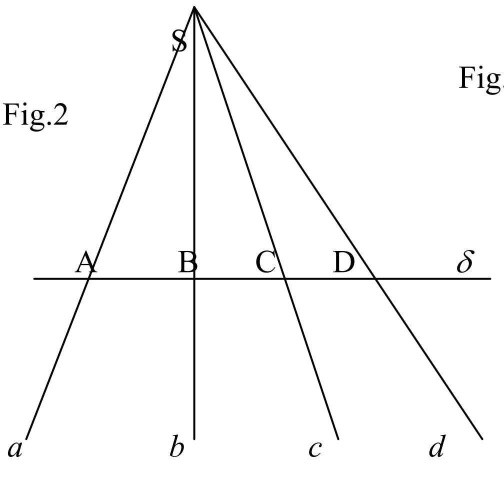
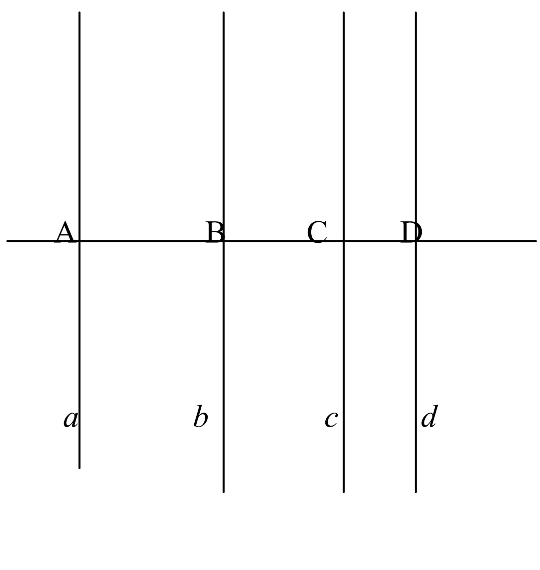


Fig.1



If the *cross-ratio* $(ABCD)\overset{def}{=}\frac{CA}{CB}:\frac{DA}{DB}$ is *harmonic* ($(ABCD)=-1$) then the pencil attachment $S(ABCD)$ is called *harmonic pencil of lines.*

2. Cross-ratio corresponding to a convergent pencil of lines

We consider the pencil of lines $S(abcd)$ cut by line δ (you see fig.1) in the points $A=\delta\cap a, B=\delta\cap b, C=\delta\cap c, D=\delta\cap d$ .If $S(XYZ)\overset{not}{=}$ triangle area with vertices X,Y and Z, $\widehat{XY}\overset{not}{=}\widehat{XSY}$, $h=d(S,\delta)$, then $\frac{CA}{CB}=\frac{CA\cdot h}{CB\cdot h}=\frac{2\cdot S(CSA)}{2\cdot S(CSB)}=\frac{S(CSA)}{S(CSB)}$.

$$(ABCD)=\frac{CA}{CB}:\frac{DA}{DB}=\frac{S(CSA)}{S(CSB)}:\frac{S(DSA)}{S(DSB)}=\frac{SC\cdot SA\cdot\sin(\widehat{ca})}{SC\cdot SB\cdot\sin(\widehat{cb})}:\frac{SD\cdot SA\cdot\sin(\widehat{da})}{SD\cdot SB\cdot\sin(\widehat{db})}=$$

$$=\frac{\sin(\widehat{ca})}{\sin(\widehat{cb})}:\frac{\sin(\widehat{da})}{\sin(\widehat{db})}.$$

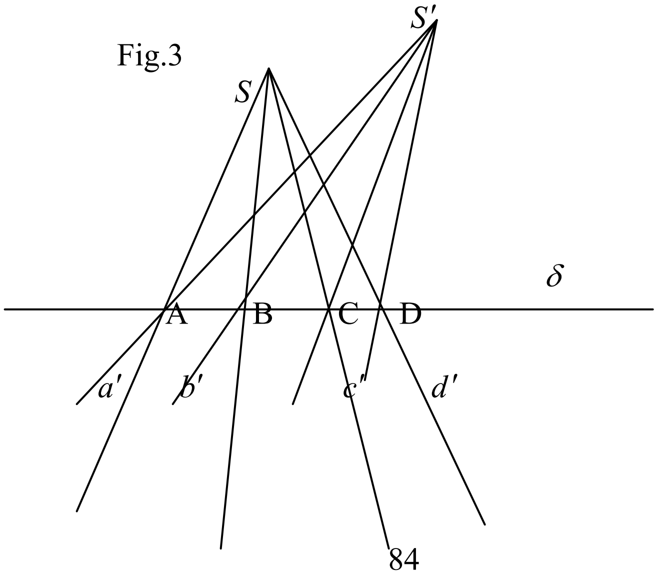
If $S(abcd)\overset{not}{=}\frac{\sin(\widehat{ca})}{\sin(\widehat{cb})}:\frac{\sin(\widehat{da})}{\sin(\widehat{db})}$, then results $(ABCD)=S(abcd)$.

3. Properties (invariant's theorems)

Theorem 1. On a line δ we consider four fixed points A,B,C,D .For any $S\notin\delta$, we denoted $a=SA, b=SB, c=SC, d=SD$. Cross-ratio corresponding to a convergent pencil of $S(abcd)$ is *invariant.*

Proof. Let $S,S'\notin\delta$, so fig.3.

Fig.3



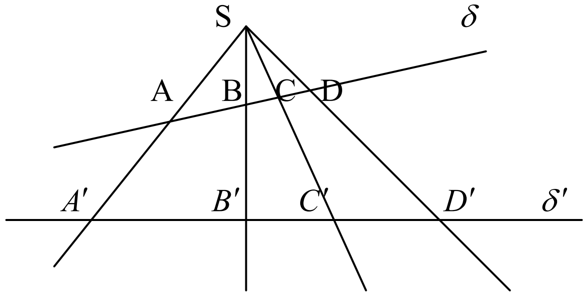
a b c d

Because $S(abcd) = (ABCD)$ and $S'(a'b'c'd') = (ABCD)$ results $S(abcd) = S'(a'b'c'd')$.(q.e.d).

Theorem 2. We consider fixed pencil of lines with vertix S and rays a,b,c,d .For any secant line δ which intersect the rays of pencil in $A = a \cap \delta, B = b \cap \delta, C = c \cap \delta,$ and $D = d \cap \delta$, Double-ratio corresponding to division $(ABCD)$ este *invariant.*

Proof. Let δ and δ' two some secant lines (you see fig.4), which intersect the rays of the pencil of lines in the points A,B,C,D and A',B',C',D' .

Fig.4

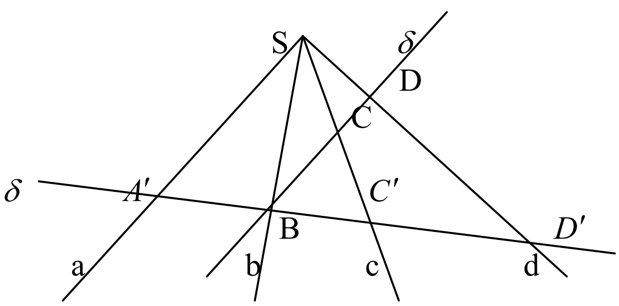


We have $(ABCD) = S(abcd)$ and $(A'B'C'D') = S(abcd)$ .Hence $(ABCD) = (A'B'C'D')$.(q.e.d.).

Pencil of lines cut by a secant paralell with one of the rays.

Let $S(abcd)$ be a pencil of lines and $\delta \| a$ (you see fig.5).

Fig.5



(1) $S(abcd) = (A'BC'D') = \frac{C'A'}{C'B} : \frac{D'A'}{D'B}$,(2) $\Delta C'A'S \approx \Delta C'BC \Rightarrow \frac{C'A'}{C'B} = \frac{SA'}{CB}$,

(3) $\Delta D'A'S \approx \Delta D'BD \Rightarrow \frac{D'A'}{D'B} = \frac{SA'}{DB}$ .Under (1),(2) and (3) results :

$S(abcd) = \frac{SA'}{CB} : \frac{SA'}{DB} = \frac{1}{CB} : \frac{1}{DB} = (A'BC'D')$.

We have the following "mnemotehnical" rule for writing the double-ratio $\frac{1}{CB}:\frac{1}{DB}$.

So $\delta \| a$ scriem $\delta \cap a = A_i$ (improperly point on the direction parallels $\delta \| a$),

$S(abcd)=\frac{CA_i}{CB}:\frac{DA_i}{DB}$ and we take $CA_i : DA_i = 1$ (switching to limit $A' \to A_i$).

$S(abcd)=\frac{1}{CB}:\frac{1}{DB}$.

Corollary. Let B,C,D be the fixed points on a line δ ,

$a\|\delta, S\in a$, $SB=b, SC=c, SD=d$.

Then $\forall S\in a, S(abcd)$ is invariant.

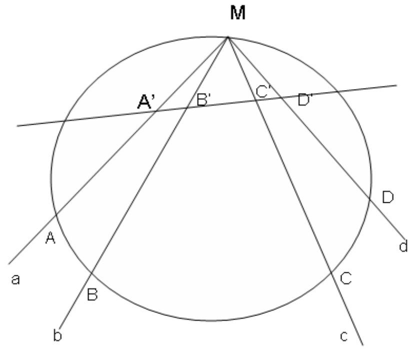
Proof. Let $A_i=\delta\cap a$. $S(abcd)=(A_iBCD)=\frac{CA_i}{CB}:\frac{DA_i}{DB}=\frac{1}{CB}:\frac{1}{DB}$=constant.

Theorem 3. Let A,B,C,D be the fixed points on $C(O;R)$ and $M\in C(O;R)$ (you see fig.6).If $MA=a, MB=b, MC=c, MD=d$ then , $\forall M\in C(O;R)$ $M(abcd)$ is invariant.

Proof. $M(abcd)=\frac{\sin(\widehat{ca})}{\sin(\widehat{cb})}:\frac{\sin(\widehat{da})}{\sin(\widehat{db})}$=constant, because A,B,C,D are fixed points and

$\widehat{ca}=\frac{\overset{\frown}{CBA}}{2R}$=ct., $\widehat{cb}=\frac{\overset{\frown}{CB}}{2R}$=ct., $\widehat{da}=\frac{\overset{\frown}{DCBA}}{2R}$=ct., $\widehat{db}=\frac{\overset{\frown}{DCB}}{2R}$=ct.

Fig.6



Observation. You see figure 6, results $M(ABCD)=M(A'B'C'D')$.

Theorem 4. Let A,B,C, D be fixed points on $C(O;R)$ and , a,b,c, d the tangents in the four points at circle $C(O;R)$ .Then whatever tangent t to the circle $C(O;R)$ in point $T\in C(O;R)$, the points $A_1=a\cap t, B_1=b\cap t, C_1=c\cap t$ şi $D_1=d\cap t$ formed a invariant division $(A_1B_1C_1D_1)$.

Proof. We have figure 7

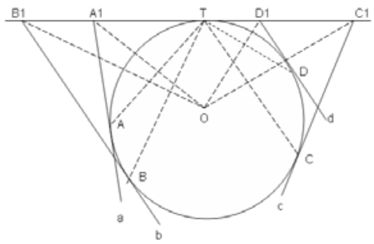


Fig.7

$(A_1B_1C_1D_1) = O(A_1B_1C_1D_1)$ .We consider the pencil of lines with vertix T and the rays $TA \perp OA_1$, $TB \perp OB_1$, $TC \perp OC_1$, $TD \perp OD_1$ .So $T(ABCD) = O(A_1B_1C_1D_1)$.

We get $(A_1B_1C_1D_1) = T(ABCD) = (\sin\frac{\overset{\frown}{CBA}}{2R} : \sin\frac{\overset{\frown}{CB}}{2R}) : (\sin\frac{\overset{\frown}{DA}}{2R} : \sin\frac{\overset{\frown}{BCD}}{2R})$ =constant.

Theorem 5. On circle $C(O;R)$ consider distinct points A,B,C,D and tangent a,b,c,d in these points at the circle (you see fig.8).We have:
$A(aBCD) = B(AbCD) = C(ABcD) = D(ABCd)$.

Proof. $A(aBCD) = (\sin\widehat{CAa} : \sin\widehat{CAB}) : (\sin\widehat{DAa} : \sin\widehat{DAB}) =$

$= (\sin\frac{\overset{\frown}{CDA}}{2R} : \sin\frac{\overset{\frown}{BC}}{2R}) : (\sin\frac{\overset{\frown}{DA}}{2R} : \sin\frac{\overset{\frown}{BCD}}{2R}) = r =$ constant=
$= B(AbCD) = C(ABcD) = D(ABCd)$ (from the equalities of sines).

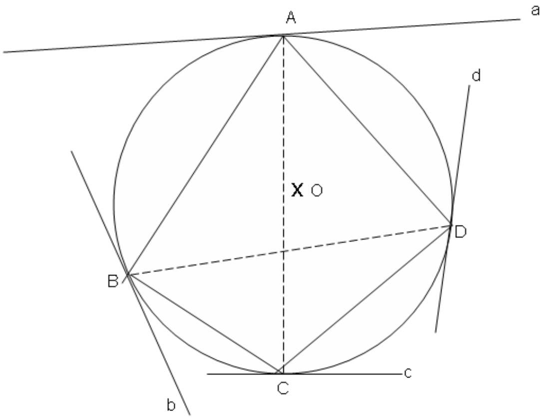


Fig. 8

Observation. *Theorem 5* represents the limit case *of the theorem 3* – the point M on the $C(O;R)$ is one of the points A,B,C or D.

Teorema 6. On circle $C(O;R)$ we consider the distinct points A,B,C,D and the tangents at the circle in these points a,b,c,d (you see fig.9).
If denoted $E = a \cap b, F = b \cap c, G = c \cap d, H = d \cap a, I = b \cap d$ and $J = a \cap c,$ then we have the equalities : $(AEJH) = (EBFI) = (JFCG) = (HIGD)$.

Proof. We consider the pencil of lines with vertex O and rays OA, OE, OJ, OH

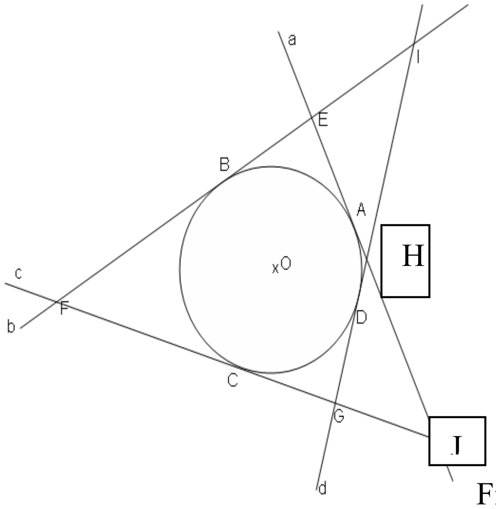


Fig.9

$OAEJH$, then the pencil of lines with vertex in A and the rays perpendiculars on the rays of previously pencil of lines : $a \perp OA, AB \perp OE, AC \perp OJ$, $AD \perp OH$, $A(aBCD)$ and we have

$(AEJH) = O(AEJH) = A(aBCD)$.

The same is obtained the equalities:

$(EBFI) = O(EBFI) = B(AbCD)$;

$(JFCG) = O(JFCG) = C(ABcD)$;

$(HIGD) = O(HIGD) = D(ABCd)$.

Now we use the equalities:

$\widehat{CAa} = \widehat{CBA} = \widehat{cCA} = \frac{\overset{\frown}{CDA}}{2R}$ şi $\widehat{CDA} = \frac{\overset{\frown}{ABC}}{2R} = \pi - \frac{\overset{\frown}{CDA}}{2R}$, and results:

$$\sin(\widehat{CAa}) = \sin(\widehat{CBA}) = \sin(\widehat{cCA}) = \sin(\widehat{CDA}) = \sin(\frac{\overset{\frown}{CDA}}{2R}) .$$

The same is obtained the equalities:

$$\sin(\widehat{CAB}) = \sin(\widehat{CBb}) = \sin(\widehat{cCB}) = \sin(\widehat{CDB}) = \sin(\frac{\overset{\frown}{CB}}{2R})$$

$$\sin(\widehat{DAa}) = \sin(\widehat{DBA}) = \sin(\widehat{DCA}) = \sin(\widehat{dDA}) = \sin(\frac{\overset{\frown}{DA}}{2R})$$

$$\sin(\widehat{DAB}) = \sin(\widehat{DBb}) = \sin(\widehat{DCB}) = \sin(\widehat{dDB}) = \sin(\frac{\overset{\frown}{DAB}}{2R}) .$$

Given these values can write:

$$A(aBCD) = B(AbCD) = C(ABcD) = D(ABCd) = (\sin\frac{\overset{\frown}{CDA}}{2R} : \sin\frac{\overset{\frown}{CB}}{2R}) : (\sin\frac{\overset{\frown}{DA}}{2R} : \sin\frac{\overset{\frown}{DAB}}{2R})$$

(q.e.d.).

Observation. *Theorem 6* represents the limit case of *theorem 4* - the tangenta t at the circle $C(O;R)$ is one of the tangents a,b,c,d .

4. Theorems on concurrence and collinearty

Theorem 7. If $(ABCD)=(AB'C'D')$ - have a *common point* A, then the lines BB', CC', DD' are concurrence.

Proof. We have the fig.10.

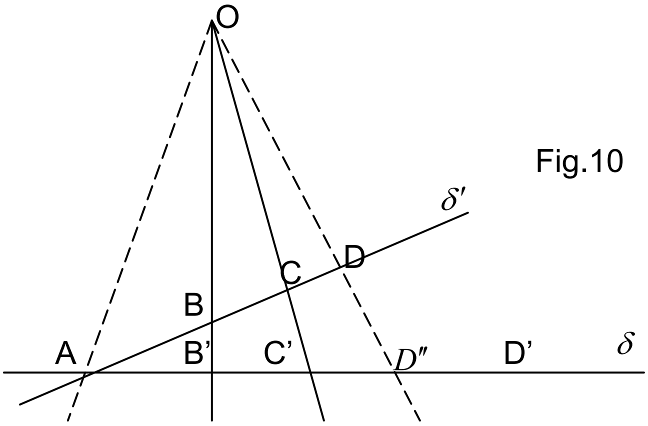


Fig.10

Let $O=BB'\cap CC'$ and $D''=OD\cap\delta'$ .We use the *theorem 2* and results $(ABCD)=(AB'C'D'')$, now use the hypothesis and we have $(ABCD)=(AB'C'D')$. Hence $(AB'C'D'')=(AB'C'D')$, then $D''=D'$.(q.e.d.).

Theorem 8. If $S(abcd)=S'(ab'c'd')$ - *common ray* $SS'=a$, then the points of intersection of the three pairs of rays correspondent: $B=b\cap b', C=c\cap c', D=d\cap d'$ are collinear.

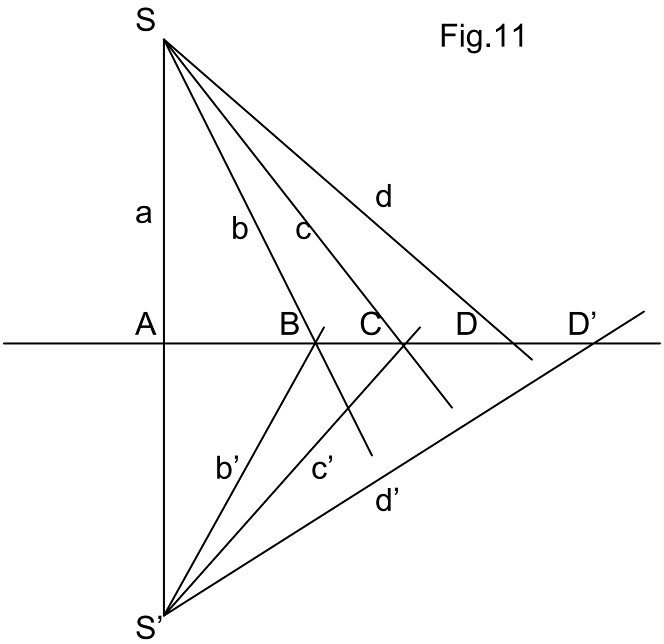


Fig.11

Proof.

Let $A=BC\cap a, D=BC\cap d$, $D'=DC\cap d'$ (fig.11).

From the hypothesis we get (1) $S(abcd)=S'(ab'c'd')$ .We intersect the pencil of lines $S(abcd)$ with BC and results (2) $S(abcd)=(ABCD)$ .We intersect the pencil of lines $S'(ab'c'd')$ with BC and results (3) $S'(ab'c'd')=(ABCD')$ .From this three relations we get $(ABCD)=(ABCD')$, then $D=D'$.(q.e.d.).

5. At the end - I propose some classical theorems that can be attacked with pencils of lines techniques*(**theorem 7 and theorem 8**)*.

Teorema 9. Pappus's theorem

If A, B and C are three points on one line , D, E and F are three points on another line , and AE meets BD at X , AF meets CD at Y , and BF meets CE at Z , then the three points X, Y and Z are collinear.

Teorema 10.Desargues's theorem.

In a projective space, two triangles are in perspective *axially* if and only if they are in perspective *centrally*.

To understand this, denote the three vertices of one triangle by (lower-case) *a*, *b*, and *c*, and those of the other by (capital) *A*, *B*, and *C*. Axial perspectivity is the condition satisfied if and only if the point of intersection of *ab* with *AB*, and that of intersection of *ac* with *AC*, and that of intersection of *bc* with *BC*, are collinear, on a line called the *axis of perspectivity*. Central perspectivity is the condition satisfied if and only if the three lines *Aa*, *Bb*, and *Cc* are concurrent, at a point called the *center of perspectivity*.

Theorem 11. Pascal's theorem (The dual of Brianchon's theorem **).**

Given a (not necessarily regular, or even convex) hexagon inscribed in a conic section, the three pairs of the continuations of opposite sides meet on a straight line, called the "Pascal line".

Theorem 12. Brianchon's theorem (The dual of Pascal's theorem)**.**

Given a hexagon circumscribed on a conic section, the lines joining opposite polygon vertices (polygon diagonals) meet in a single point.

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Neculai Stanciu
Department of mathematics
High School "Saint Mc. Sava" Berca
Micro V, Bl. 36, Ap. 15, Buzău, Romania
stanciuneculai@yahoo.com

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Neculai STANCIU (CV)

Profesor de matematicã

Studii
Master în management educaţional şi comunicare instituţionalã, SNSPA, Bucureşti (2005 – 2007)
Licenţiat în matematicã, Universitatea Bucureşti (1992 – 1998)
Licenţiat al Facultãţii de Tehnologia Construcţiilor de Maşini, Universitatea Tehnicã ''Gh. Asachi'' Iaşi (1987 – 1992)
Cursuri Postuniversitare pentru profesionalizarea pedagogicã a absolvenţilor de învăţământ superior, Universitatea Politehnicã Bucureşti, (1994)

Experienţă profesionalã
Titular la Şc."GeorgeEmil Palade" şi Şc. Nr.6 – Buzău (din 2009)
Titular la Grupul Şcolar Tehnic ''Sf. Mucenic Sava '' , Berca, Buzãu (1997 – 2009)
Lider de sindicat la Grupul Şcolar Tehnic ''Sf. Mucenic Sava '' , Berca, Buzãu (2004 – 2005)
Director adjunct la Grupul Şcolar Tehnic ''Sf. Mucenic Sava '' , Berca, Buzãu (2005)
Director la Grupul Şcolar Tehnic ''Sf. Mucenic Sava '' , Berca, Buzãu (din 2006)
Lider de sindicat la Grupul Şcolar Tehnic ''Sf. Mucenic Sava '' , Berca, Buzãu (2008)
Şeful catedrei de matematică (2008 – 2009)
Titular la Şcoala George Emil Palade Buzău şi Şcoala nr. 6 Buzău (2009 – 2010)

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