

Mayhem Solutions

M 345. Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

The area of isosceles $\triangle ABC$ is $q\sqrt{15}$. Given that $AB = 2BC$, express the perimeter of $\triangle ABC$ in terms of q .

Solution by Neculai Stanciu, High School "Saint Sava", Berca, Romania.

Obviously $q > 0$.

We have two cases.

1. Case. $AB = AC > BC$.

$AB = AC = 2a$, $BC = a$, and the perimeter is $P = 5a$.

With Heron's formula we have $S = \sqrt{\frac{5a}{2} \cdot \left(\frac{a}{2}\right) \cdot \left(\frac{a}{2}\right) \cdot \left(\frac{3a}{2}\right)} = \frac{a^2\sqrt{15}}{4} \stackrel{\text{the hypothesis}}{=} q\sqrt{15}$.

Results $a = 2\sqrt{q}$ and the perimeter $P = 10\sqrt{q}$.

2. Case. $AC = BC = a < AB = 2a$.

We do in the same way as 1. Case, and obtain $P = 4a$, $S = \sqrt{2a \cdot (a) \cdot (a) \cdot (0)} = 0 \stackrel{\text{the hypothesis}}{=} q\sqrt{15}$

.False.

In conclusion we have the solutions $P = 10\sqrt{q}$

M 349. Proposed by the Mayhem Staff.

(a) Find all ordered pairs of integers (x,y) with $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) How many ordered pairs of integers (x,y) are there with $\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}$?

Solution by Neculai Stanciu, High School "Saint Sava", Berca, Romania

$$(a) \frac{1}{x} + \frac{1}{y} = \frac{1}{5} \Leftrightarrow 5y + 5x = xy \Leftrightarrow 5y = x(y-5) \Leftrightarrow x = \frac{5y}{y-5}$$

$$x = \frac{5y}{y-5} \in \mathbb{Z} \Rightarrow \begin{cases} y-5 \mid 5y \\ y-5 \mid y-5 \stackrel{\times(-5)}{\Rightarrow} y-5 \mid -5y+25 \Rightarrow y-5 \mid 25 \end{cases}$$

$$\Rightarrow y-5 \in D_{25} = \{\pm 1, \pm 5, \pm 25\} \Rightarrow (1) y \in \{6, 10, 30, 4, 0, -20\}.$$

$$x = \frac{5y}{y-5} \Rightarrow (y, x) \in \{(6, 30), (10, 10), (30, 6), (4, -20), (0, 0), (-20, 4)\}.$$

The pair $(0,0)$ is not a solution. So he have 5(five) solutions

$$(y, x) \in \{(6, 30), (10, 10), (30, 6), (4, -20), (-20, 4)\}.$$

(b) We do in the same way as (a) point .

We get $y - 1200 \mid (1200)^2$. $(1200)^2 = 2^8 \cdot 3^2 \cdot 5^4$, so he have $(8+1)(2+1)(4+1) = 9 \cdot 3 \cdot 5$

naturals divisors and $2 \cdot (9 \cdot 3 \cdot 5) = 270$ integers divisors of $(1200)^2$.

If you eliminate the solution $(0,0)$ there are so, 269 pairs of integers (x,y) are there with $\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}$.

M 357. Proposed by Mayhem Staff.

Determine all real numbers x that satisfy $3^{2x+2} + 3 = 3^x + 3^{x+3}$.

Solution by Neculai Stanciu, High School "Saint Sava", Berca, Romania.

$3^{2x+2} + 3 = 3^x + 3^{x+3} \Leftrightarrow (3^x)^2 \cdot 9 + 3 = 3^x + 3^x \cdot 27$, we denoted by $3^x = t$, and results $9t^2 - 28t + 3 = 0$.

$$\Delta = 28^2 - 4 \cdot 3 \cdot 9 = 676, t_{1,2} = \frac{28 \pm \sqrt{676}}{18} = \frac{28 \pm 26}{18}, t_1 = 3, t_2 = \frac{1}{9}.$$

We return to the substitution and obtain $x_1 = 3, x_2 = -2$.

$$x \in \{-2, 3\}.$$

Mayhem Solutions

M 357. Proposed by George Tsapakides, Agrinio, Greece.

Let a, b and c be positive real numbers. Prove that

$$ab(a + b - c) + bc(b + c - a) + ca(c + a - b) \geq 3abc.$$

Solution by Neculai Stanciu, High School "Saint Sava", Berca, Romania.

We use the Theorem: $\frac{x}{y} + \frac{y}{x} \geq 2, \forall x, y \geq 0$.

Proof. $\frac{x}{y} + \frac{y}{x} \geq 2 \Leftrightarrow (x - y)^2 \geq 0$. Q.E.D.

We divide the equation $ab(a+b-c)+bc(b+c-a)+ca(c+a-b)\geq 3abc$, with abc , and obtain

$$\frac{a+b-c}{c} + \frac{b+c-a}{a} + \frac{c+a-b}{b} \geq 3 \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{c} + \frac{b}{c} - 1 + \frac{b}{a} + \frac{c}{a} - 1 + \frac{c}{b} + \frac{a}{b} - 1 \geq 2 + 2 + 2 - 3 = 3. \text{Q.E.D.}$$

Mayhem Solutions

M 357. Proposed by Mayhem Staff.

Solution by Neculai Stanciu, High School "Saint Sava", Berca, Romania.

We denoted by: $C = \text{circle}$, $DE \cap C = P$, $EF \cap C = N$, $FG \cap C = M$, $GD \cap C = Q$, $GQ = GM = x$,

$$H = \text{Pr}_{GF} D, GH = x - 1, GD = x + 1, DH = 4.$$

$$GH^2 + DH^2 = GD^2 \Rightarrow x = 4.$$

$$EF = 2 \cdot \text{radius} = 4.$$

$$GF = GH + HF = 6$$

$$\text{Area } DEFG = \frac{(DE + GF) \cdot EF}{2} = 18$$