A NOTE ON THE DROZ – FARNY LINE

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In 1899, Arnold Droz-Farny (1865-1912), a Swiss science and mathematics teacher, published without proof the following

Theorem 1. If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the three sidelines. The midpoints of the three segments are collinear.

Many authors have dealt with this assertion (see below) and [4] contains, once more!, without proof the following generalization, whose proof is the object of this note:

Theorem 2. Given triangle ABC with orthocenter H. A rectangular hyperbola with center H intersects line BC in A_1 and A_2, line CA in B_1 and B_2 and line AB in C_1 and C_2. Prove that the points P, Q and R, the midpoints of A_1A_2, B_1B_2 and C_1C_2, respectively, are collinear.

Proof : Suppose that the asymptotes of the hyperbola remain fixed and the triangle rotate around the orthocenter. Choose a complex system of coordinates with the origin O in the circumcircle of the triangle ABC, the asymptotes of the rectangular hyperbola parallel to the real and imaginary axes and the vertices of the triangle on the unit circle. It follows that $A(a)$, $B(b)$, $C(c)$ and $H(h)$, with $a, b, c, h \in \mathbb{C}$, $|a| = |b| = |c| = 1$ and $h = a + b + c$. After the change of coordinates $z \mapsto -z + h$ (a translation followed by a point reflection), the asymptotes become axes of coordinates and the coordinates of the points are $A(b + c)$, $B(c + a)$, $C(A + b)$ and $H(0)$. It is not necessary to rotate the triangle around the orthocenter, because the vertices have been arbitrarily chosen.

In Cartesian coordinates, the equation of a rectangular hyperbola having the axes of coordinates as asymptotes is $xy = m$, with m a real nonzero constant, so that the hyperbola has in complex coordinates the equation

$$z^2 - \bar{z}^2 = n, \text{ with } n \in \mathbb{C} - \mathbb{R}. \tag{1}$$

Line BC has the equation

$$z + bc\bar{z} = a + b + c + \frac{bc}{a}. \tag{2}$$

From (1) and (2) we obtain, by the elimination of $\bar{z}$, the quadratic equation

$$(b^2c^2 - 1)z^2 + 2\left(a + b + c + \frac{bc}{a}\right)z - \left(a + b + c + \frac{bc}{a}\right)^2 - b^2c^2n = 0, \tag{3}$$

whose solutions are a_1 and a_2, the coordinates of the points A_1 and A_2 respectively. It follows that the coordinate p of the point P is

$$p = \frac{a_1+a_2}{2} = \frac{a+b+c+\frac{bc}{a}}{1-b^2c^2} \tag{4}$$

and we can obtain, by the same way, q and r, the coordinates of the points Q and R respectively.

The points P, Q, R are collinear if and only if the following determinant vanishes

$$\Delta = \begin{vmatrix} p & \bar{p} & 1 \\ q & \bar{q} & 1 \\ r & \bar{r} & 1 \end{vmatrix} = 0. \tag{5}$$

Taking account of the particular form of the coordinates p, q, r, we can write the determinant Δ as a sum of four determinants

$$\Delta = |a+b+c|^2\Delta_1 + abc(a+b+c)\Delta_2 + a^2b^2c^2(ab+bc+ca)\Delta_3 + a^2b^2c^2\Delta_4, \tag{6}$$

where

$$\Delta_1 = \begin{vmatrix} \frac{1}{1-b^2c^2} & \frac{-b^2c^2}{1-b^2c^2} & 1 \\ \frac{1}{1-c^2a^2} & \frac{-c^2a^2}{1-c^2a^2} & 1 \\ \frac{1}{1-a^2b^2} & \frac{-a^2b^2}{1-a^2b^2} & 1 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \frac{1}{1-b^2c^2} & \frac{-1}{1-b^2c^2} & 1 \\ \frac{1}{1-c^2a^2} & \frac{-1}{1-c^2a^2} & 1 \\ \frac{1}{1-a^2b^2} & \frac{-1}{1-a^2b^2} & 1 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} \frac{1}{a^2(1-b^2c^2)} & \frac{-1}{a^2(1-b^2c^2)} & 1 \\ \frac{1}{b^2(1-c^2a^2)} & \frac{-1}{b^2(1-c^2a^2)} & 1 \\ \frac{1}{c^2(1-a^2b^2)} & \frac{-1}{c^2(1-a^2b^2)} & 1 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} \frac{1}{a^2(1-b^2c^2)} & \frac{-1}{1-b^2c^2} & 1 \\ \frac{1}{b^2(1-c^2a^2)} & \frac{-1}{1-c^2a^2} & 1 \\ \frac{1}{c^2(1-a^2b^2)} & \frac{-1}{1-a^2b^2} & 1 \end{vmatrix}.$$

The sum of the first two columns equals the third in Δ_1 and equals zero in Δ_2 and Δ_3, so that $\Delta_1 = \Delta_2 = \Delta_3 = 0$. Direct calculations (is easy!) show that $\Delta_4 = 0$.

We substitute these values in (6) and we obtain (5), which ends the proof of the Theorem 2.

For $n = 0$ the hyperbola degenerates in two perpendicular straight lines (the asymptotes) and we get a proof of the Theorem 1, since the points P, Q, R are the same.

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