ON A PROBLEM OF DIEUDONNÉ

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In the book [1] Jean Dieudonné has proposed the following problem :

Let u_0 be an arbitrary number in $(0, \pi)$ and $u_{n+1} = \sin u_n$, for $n = 0, 1, 2, \ldots$. Compute $\lim_{n\to\infty} \sqrt{n}u_n$.

The solution of this problem is presented in [2] (page 487). The same method allow us to obtain the following generalization :

Let $a > 0$ be a real number and the function $f : [0. a) \to [0, a)$ having the following properties :

a) f has derivatives of each order.

b) $f(0) = f''(0) = 0$, $f'(0) = 1$, $f'''(0) < 0$.

c) $f(x) < x$, for every $x \in (0, a)$.

Compute $\lim_{n\to\infty} \sqrt{n}u_n$, where $u_{n+1} = f(u_n)$ and $u_0 \in (0, a)$ is arbitrary.

We shall consider known the following property of the numerical sequences : the convergence in the sense of Cauchy implies the convergence in the sense of Cesaro to the same limit, namely

$$\text{if } \lim_{n\to\infty} a_n = l, \text{ then } \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = l.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded, $0 < u_n < a$, for every $n \in \mathbb{N}$. The sequence is decreasing, since $u_{n+1} = f(u_n) < u_n$, for every $n \in \mathbb{N}$. By the theorem of *Weierstrass*, the sequence converges to a limit l, such that $l = f(l)$. The conditions from the hypothesis imply that $l = 0$. This is a necessary condition for $\lim_{n\to\infty} \sqrt{n}u_n$ to be finite.

We apply the convergence property for the sequence $a_n = \dfrac{1}{u_{n+1}^2} - \dfrac{1}{u_n^2}$ and it follows that

$$\lim_{n\to\infty}\left(\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{1}{u_{k+1}^2} - \frac{1}{u_k^2}\right) = \lim_{n\to\infty}\frac{1}{nu_n^2}. \tag{1}$$

We have

$$\lim_{n\to\infty}\left(\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{f^2(u_n)} - \frac{1}{u_n^2}\right) = \lim_{x\to 0}\left(\frac{1}{f^2(x)} - \frac{1}{x^2}\right). \tag{2}$$

For the calculation of the last limit we apply the *l'Hospital*'s rule :

$$\lim_{x\to 0}\frac{x^2}{f^2(x)} = \left(\lim_{x\to 0}\frac{x}{f(x)}\right)^2 = \left(\lim_{x\to 0}\frac{1}{f'(x)}\right)^2 = \left(\frac{1}{f'(0)}\right)^2 = 1, \tag{3}$$

$$\lim_{x\to 0}\frac{x+f(x)}{x} = \lim_{x\to 0}\left(1 + f'(x)\right) = 1 + f'(0) = 2, \tag{4}$$

$$\lim_{x\to 0}\frac{x - f(x)}{x^3} = \lim_{x\to 0}\frac{1 - f'(x)}{3x^2} = \lim_{x\to 0}\frac{-f''(x)}{6x} = \lim_{x\to 0}\frac{-f'''(x)}{6} = \frac{-f'''(0)}{6}. \tag{5}$$

From the equalities (1) − (5) we deduce

$$\lim_{n\to\infty}\sqrt{n}u_n = \sqrt{1\cdot\frac{1}{2}\cdot\frac{-6}{f'''(0)}} = \sqrt{\frac{-3}{f'''(0)}}.$$

Notes : 1) The limit (2) for the function sinus is obtained in [2] using the expansion in *Taylor* series in a neighborhood of the origin.

2) If $f(0) = f''(0) = 0$, a limit similar to (5) is

$$\lim_{x\to 0}\frac{xf'(x) - f(x)}{x^3} = \lim_{x\to 0}\frac{f'(x) + xf''(x) - f'(x)}{3x^2} = \lim_{x\to 0}\frac{f''(x)}{3x} = \lim_{x\to 0}\frac{f'''(x)}{3} = \frac{f'''(0)}{3}.$$

For $f(x) = \sin x$, this limit is given in [2], page 398.

3) If $f'(0) \neq 0$, $f''(0) = 0$ and $f(x) < xf'(0) + f(0)$, then the function f can be replaced by the function g, defined by $g(x) = \dfrac{f(x) - f(0)}{f'(0)}$.

References

1. Dieudonné, J., Infinitesimal calculus, Hermann, Paris, 1962 (in French)

2. (editor) Teodorescu, N., Problems from *The Mathematical Gazette*, Technical Publishing House, Bucharest, 1984 (in Romanian)

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