

Concurrence and collinearity using properties of pencils of lines

“Projective geometry is whole geometry”
Arthur Cayley

Neculai Stanciu¹

Abstract.

This article is devoted to the study of two fundamental and reciprocal questions: when do three given points lie on a single line, and when do three given lines pass through a single point? The techniques we describe in this article will be augmented by more sophisticated approaches, such as the Pappus’s theorems, the Desargues’s theorems, the Pascal’s theorem and the Brianchon’s theorem.

The formalism of projective geometry makes a discussion of such properties possible, and exposes some remarkable facts, such as the duality of points and lines. While technique “cross-ratio” of four points, and in the light of duality the cross-ratio of four lines can be useful on contest problems, much of the material here is considered “too advanced” for primary and secondary school education. This is a pity, as some of the most beautiful classical geometry appears only in the projective geometry.

1. Main purpose - of the results below is familiarizing readers with new methods (*little known even teachers of mathematics*) solving problems of concurrence and collinearity namely the techniques offered by pencils of lines properties.

We consider fig.1 where $S(a,b,c,d)$ or $S(A,B,C,D)$ represents a *convergent pencil of lines*, with its own point S and rays a,b,c,d or SA,SB,SC,SD and fig.2 where $S(a,b,c,d)$ is a *parallel pencil of lines* with rays a,b,c,d or SA,SB,SC,SD (S is *improperly point*).

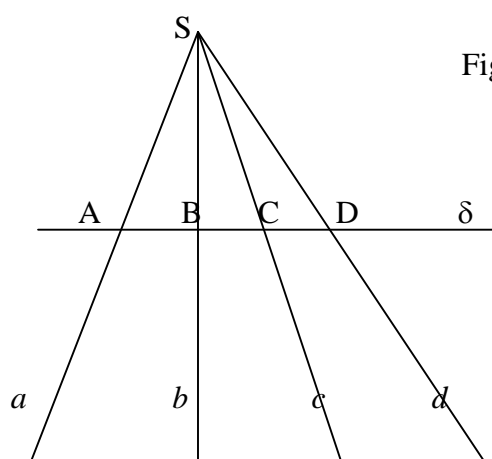


Fig.1

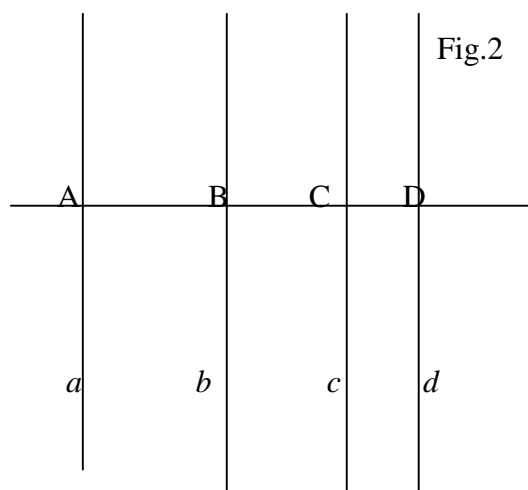


Fig.2

¹ Prof. , „George Emil Palade” Secondary School, Buz u, Romania;
e-mail:stanciuneculai@yahoo.com

If the *cross-ratio* $(ABCD) \stackrel{def}{=} \frac{CA}{CB} : \frac{DA}{DB}$ is *harmonic* $((ABCD) = -1)$ then the pencil attachment $S(ABCD)$ is called *harmonic pencil of lines*.

2. Cross-ratio corresponding to a convergent pencil of lines

We consider the pencil of lines $S(abcd)$ cut by line δ (you see fig.1) in the points

$A = \delta \cap a, B = \delta \cap b, C = \delta \cap c, D = \delta \cap d$. If $S(XYZ) \stackrel{not}{=} \text{triangle area with vertices } X, Y \text{ and } Z$, $\hat{XY} \stackrel{not}{=} \hat{XSY}$, $h = d(S, \delta)$, then $\frac{CA}{CB} = \frac{CA \cdot h}{CB \cdot h} = \frac{2 \cdot S(CSA)}{2 \cdot S(CSB)} = \frac{S(CSA)}{S(CSB)}$.

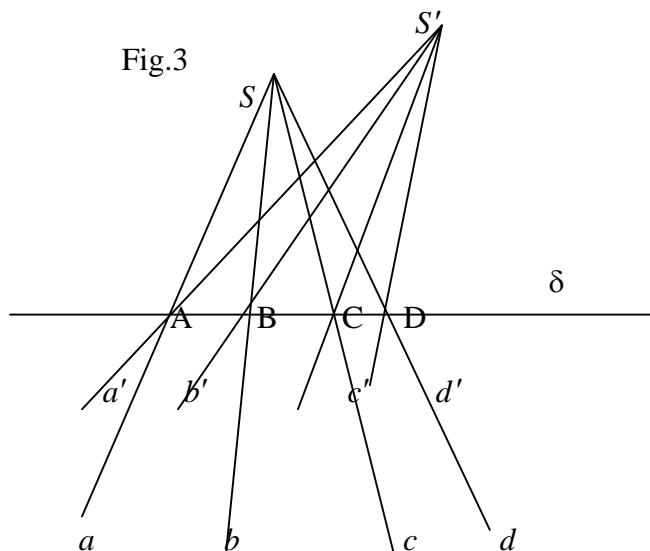
$$(ABCD) = \frac{CA}{CB} : \frac{DA}{DB} = \frac{S(CSA)}{S(CSB)} : \frac{S(DSA)}{S(DSB)} = \frac{SC \cdot SA \cdot \sin(\hat{ca})}{SC \cdot SB \cdot \sin(\hat{cb})} : \frac{SD \cdot SA \cdot \sin(\hat{da})}{SD \cdot SB \cdot \sin(\hat{db})} = \frac{\sin(\hat{ca})}{\sin(\hat{cb})} : \frac{\sin(\hat{da})}{\sin(\hat{db})}$$

If $S(abcd) \stackrel{not}{=} \frac{\sin(\hat{ca})}{\sin(\hat{cb})} : \frac{\sin(\hat{da})}{\sin(\hat{db})}$, then results $(ABCD) = S(abcd)$.

3. Properties (invariant's theorems)

Theorem 1. On a line δ we consider four fixed points A, B, C, D . For any $S \notin \delta$, we denoted $a = SA, b = SB, c = SC, d = SD$. Cross-ratio corresponding to a convergent pencil of $S(abcd)$ is *invariant*.

Proof. Let $S, S' \notin \delta$, so fig.3.

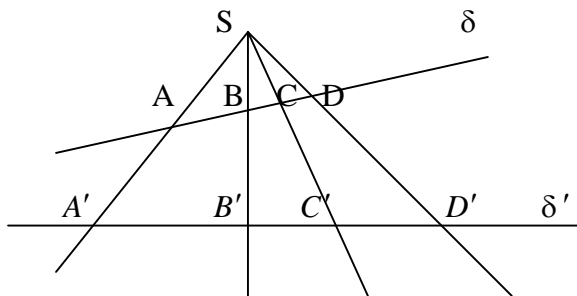


Because $S(abcd) = (ABCD)$ and $S'(a'b'c'd') = (ABCD)$ results $S(abcd) = S'(a'b'c'd')$. (q.e.d).

Theorem 2. We consider fixed pencil of lines with vertex S and rays a, b, c, d . For any secant line δ which intersect the rays of pencil in $A = a \cap \delta, B = b \cap \delta, C = c \cap \delta,$ and $D = d \cap \delta$, Double-ratio corresponding to division $(ABCD)$ este *invariant*.

Proof. Let δ and δ' two some secant lines (you see fig.4), which intersect the rays of the pencil of lines in the points A, B, C, D and A', B', C', D' .

Fig.4

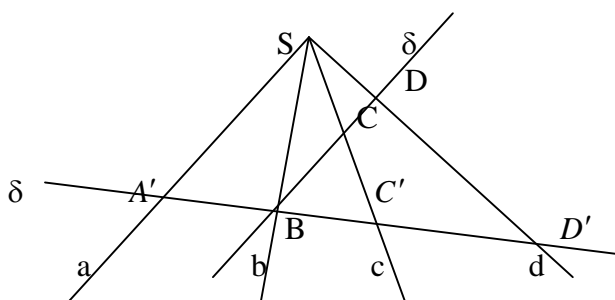


We have $(ABCD) = S(abcd)$ and $(A'B'C'D') = S(abcd)$. Hence $(ABCD) = (A'B'C'D')$. (q.e.d.).

Pencil of lines cut by a secant paralell with one of the rays.

Let $S(abcd)$ be a pencil of lines and $\delta \parallel a$ (you see fig.5).

Fig.5



$$(1) S(abcd) = (A'BC'D') = \frac{C'A'}{C'B} : \frac{D'A'}{D'B}, (2) \Delta C'A'S \approx \Delta C'BC \Rightarrow \frac{C'A'}{C'B} = \frac{SA'}{CB},$$

$$(3) \Delta D'A'S \approx \Delta D'BD \Rightarrow \frac{D'A'}{D'B} = \frac{SA'}{DB}. \text{ Under (1),(2) and (3) results :}$$

$$S(abcd) = \frac{SA'}{CB} : \frac{SA'}{DB} = \frac{1}{CB} : \frac{1}{DB} = (A'BC'D').$$

We have the following “mnemotekhnical” rule for writing the double-ratio $\frac{1}{CB} : \frac{1}{DB}$.

So $\delta \parallel a$ scriem $\delta \cap a = A_i$ (improperly point on the direction parallels $\delta \parallel a$),

$$S(abcd) = \frac{CA_i}{CB} : \frac{DA_i}{DB} \text{ and we take } CA_i : DA_i = 1 \text{ (switching to limit } A' \rightarrow A_i).$$

$$S(abcd) = \frac{1}{CB} : \frac{1}{DB} .$$

Corollary. Let B, C, D be the fixed points on a line δ , $a \parallel \delta, S \in a, SB = b, SC = c, SD = d$. Then $\forall S \in a, S(abcd)$ is invariant.

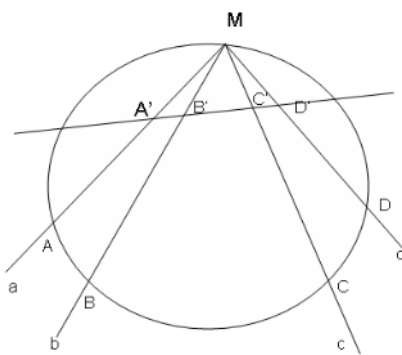
Proof. Let $A_i = \delta \cap a$. $S(abcd) = (A_i BCD) = \frac{CA_i}{CB} : \frac{DA_i}{DB} = \frac{1}{CB} : \frac{1}{DB} = \text{constant}$.

Theorem 3. Let A, B, C, D be the fixed points on $C(O; R)$ and $M \in C(O; R)$ (you see fig.6). If $MA = a, MB = b, MC = c, MD = d$ then, $\forall M \in C(O; R)$ $M(abcd)$ is invariant.

Proof. $M(abcd) = \frac{\sin(\widehat{ca})}{\sin(\widehat{cb})} : \frac{\sin(\widehat{da})}{\sin(\widehat{db})} = \text{constant}$, because A, B, C, D are fixed points and

$$\widehat{ca} = \frac{\widehat{CBA}}{2R} = \text{ct.}, \widehat{cb} = \frac{\widehat{CB}}{2R} = \text{ct.}, \widehat{da} = \frac{\widehat{DCBA}}{2R} = \text{ct.}, \widehat{db} = \frac{\widehat{DCB}}{2R} = \text{ct.}$$

Fig.6



Observation. You see figure 6, results $M(ABCD) = M(A'B'C'D')$.

Theorem 4. Let A, B, C, D be fixed points on $C(O; R)$ and a, b, c, d the tangents in the four points at circle $C(O; R)$. Then whatever tangent t to the circle $C(O; R)$ in point $T \in C(O; R)$, the points $A_1 = a \cap t, B_1 = b \cap t, C_1 = c \cap t$ i $D_1 = d \cap t$ formed a invariant division $(A_1 B_1 C_1 D_1)$.

Proof. We have figure 7

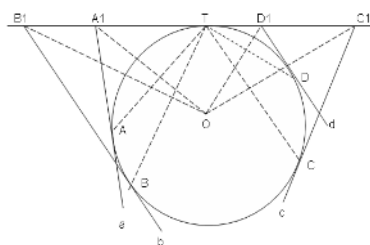


Fig.7

$(A_1B_1C_1D_1) = O(A_1B_1C_1D_1)$. We consider the pencil of lines with vertex T and the rays $TA \perp OA_1, TB \perp OB_1, TC \perp OC_1, TD \perp OD_1$. So $T(ABCD) = O(A_1B_1C_1D_1)$.

We get $(A_1B_1C_1D_1) = T(ABCD) = (\sin \frac{\widehat{CBA}}{2R} : \sin \frac{\widehat{CB}}{2R}) : (\sin \frac{\widehat{DA}}{2R} : \sin \frac{\widehat{BCD}}{2R}) = \text{constant}$.

Theorem 5. On circle $C(O;R)$ consider distinct points A, B, C, D and tangent a, b, c, d in these points at the circle (you see fig.8). We have:

$$A(aBCD) = B(AbCD) = C(ABcD) = D(ABCd).$$

Proof. $A(aBCD) = (\sin \widehat{CAa} : \sin \widehat{CAB}) : (\sin \widehat{DAa} : \sin \widehat{DAB}) =$
 $= (\sin \frac{\widehat{CDA}}{2R} : \sin \frac{\widehat{BC}}{2R}) : (\sin \frac{\widehat{DA}}{2R} : \sin \frac{\widehat{BCD}}{2R}) = r = \text{constant} =$
 $= B(AbCD) = C(ABcD) = D(ABCd)$ (from the equalities of sines).

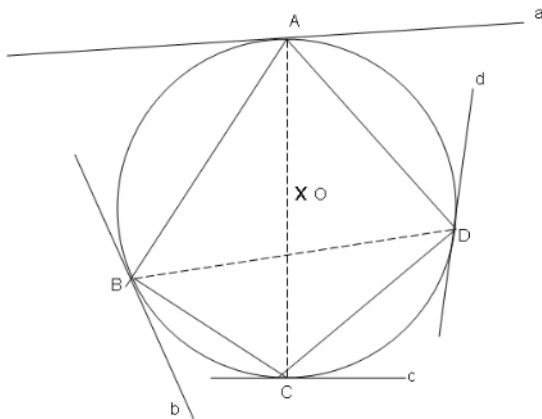


Fig. 8

Observation. Theorem 5 represents the limit case of the theorem 3 – the point M on the $C(O;R)$ is one of the points A, B, C or D .

Teorema 6. On circle $C(O;R)$ we consider the distinct points A, B, C, D and the tangents at the circle in these points a, b, c, d (you see fig.9).

If denoted $E = a \cap b, F = b \cap c, G = c \cap d, H = d \cap a, I = b \cap d$ and $J = a \cap c$, then we have the equalities : $(AEJH) = (EBFI) = (JFCG) = (HIGD)$.

Proof. We consider the pencil of lines with vertex O and rays OA, OE, OJ, OH

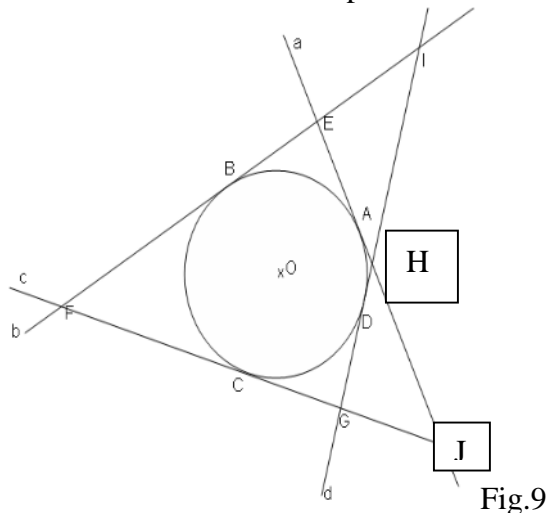


Fig.9

$OAEJH$, then the pencil of lines with vertex in A and the rays perpendicular on the rays of previously pencil of lines : $a \perp OA, AB \perp OE, AC \perp OJ, AD \perp OH, A(aBCD)$ and we have

$$(AEJH) = O(AEJH) = A(aBCD).$$

The same is obtained the equalities:

$$(EBFI) = O(EBFI) = B(ABcD);$$

$$(JFCG) = O(JFCG) = C(ABcD);$$

$$(HIGD) = O(HIGD) = D(ABCd).$$

Now we use the equalities:

$$\hat{C}Aa = \hat{C}Ba = \hat{c}CA = \frac{\widehat{CDA}}{2R} \quad \text{și} \quad \widehat{CDA} = \frac{\widehat{ABC}}{2R} = \pi - \frac{\widehat{CDA}}{2R}, \text{ and results:}$$

$$\sin(\hat{C}Aa) = \sin(\hat{C}Ba) = \sin(\hat{c}CA) = \sin(\widehat{CDA}) = \sin\left(\frac{\widehat{CDA}}{2R}\right).$$

The same is obtained the equalities:

$$\sin(\hat{C}AB) = \sin(\hat{C}Bb) = \sin(\hat{c}CB) = \sin(\widehat{CDB}) = \sin\left(\frac{\widehat{CB}}{2R}\right)$$

$$\sin(\hat{D}Aa) = \sin(\hat{D}Ba) = \sin(\hat{D}CA) = \sin(\widehat{dDA}) = \sin\left(\frac{\widehat{DA}}{2R}\right)$$

$$\sin(\hat{D}AB) = \sin(\hat{D}Bb) = \sin(\hat{D}CB) = \sin(\widehat{dDB}) = \sin\left(\frac{\widehat{DAB}}{2R}\right).$$

Given these values can write:

$$A(aBCD) = B(ABcD) = C(ABcD) = D(ABCd) = \left(\sin \frac{\widehat{CDA}}{2R} : \sin \frac{\widehat{CB}}{2R}\right) : \left(\sin \frac{\widehat{DA}}{2R} : \sin \frac{\widehat{DAB}}{2R}\right)$$

(q.e.d.).

Observation. *Theorem 6* represents the limit case of *theorem 4* - the tangenta t at the circle $C(O; R)$ is one of the tangents a, b, c, d .

4. Theorems on concurrence and collinearty

Theorem 7. If $(ABCD) = (AB'C'D')$ - have a *common point* A , then the lines BB', CC', DD' are concurrence.

Proof. We have the fig.10.

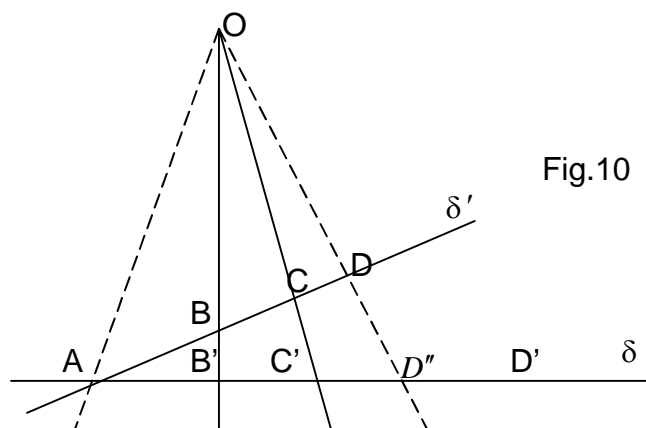


Fig.10

Let $O = BB' \cap CC'$ and $D'' = OD \cap \delta'$. We use the *theorem 2* and results $(ABCD) = (AB'C'D'')$, now use the hypothesis and we have $(ABCD) = (AB'C'D')$. Hence $(AB'C'D'') = (AB'C'D')$, then $D'' = D'$. (q.e.d.).

Theorem 8. If $S(abcd) = S'(ab'c'd')$ - *common ray* $SS' = a$, then the points of intersection of the three pairs of rays correspondent: $B = b \cap b', C = c \cap c', D = d \cap d'$ are collinear.

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