UN CAZ DE REZOLVARE AL ECUATIILOR DE GRAD 7

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Fie un polinom de grad 7 de forma:

$$f\in\mathbf{R[X]}, f=X^7+a_1X^5+a_2X^3+a_3X+a_4 \text{ cu } a_1, a_2, a_3, a_4\in\mathbf{R} \quad \mathbf{(1)}$$

Ne propunem sa aflam in ce conditii ecuatia algebrica f(x)=0 poate fi “rezolvabila prin radicali”,si mai mult, sa rezolvam aceasta ecuatie.

Observatia 1

Se stie ca orice polinom de grad impar cu coeficienti reali admite cel putin o radacina reala.

Observatia 2

Vom presupune $a_4\neq 0$,cautand solutii reale nenule ale ecuatiei f(x)=0 .

Proprietatea 1

Ecuatia f(x)=0 cu f avand forma (1) poate fi rezolvabila prin radicali ,daca au loc relatiile:

$$7a_2=2a_1^2 \text{ si } 49a_3=a_1^3 \qquad (2)$$

Solutia 1

Cautam solutii reale de forma $x=u+v$, $u,v\in\mathbf{R}$. Notam uv=b si folosim urmatoarele identitati:

$$(u+v)^7=u^7+v^7+7uv(u^5+v^5)+21u^2v^2(u^3+v^3)+35u^3v^3(u+v)$$
$$(u+v)^5=u^5+v^5+5uv(u^3+v^3)+10u^2v^2(u+v)$$

Deducem: $u^5+v^5=x^5-[5b(x^3-3xb)+10b^2x]$

Respectiv: $u^7+v^7=x^7-7b(x^5-5bx^3+15b^2x-10b^2x)-21b^2(x^3-3xb)-35b^3x=$
$=x^7-7bx^5+14b^2x^3-7b^3x$

Asadar : $\begin{cases} u^7+v^7=x^7-7bx^5+14b^2x^3-7b^3x \\ u^7v^7=b^7 \end{cases}$

Numerele u^7 , v^7 pot fi privite ca fiind solutii ale ecuatiei:

$$t^2-(x^7-7bx^5+14b^2x^3-7b^3x)t+b^7=0$$

Notam $x^7-7bx^5+14b^2x^3-7b^3x = -C$ si obtinem :

$$u^7=\frac{-C+\sqrt{C^2-4b^7}}{2} \quad , v^7=\frac{-C-\sqrt{C^2-4b^7}}{2}$$

In consecinta , numarul de forma:

$$x=\sqrt[7]{\frac{-C+\sqrt{C^2-4b^7}}{2}}+\sqrt[7]{\frac{-C-\sqrt{C^2-4b^7}}{2}}$$

este solutie a ecuatiei: (3) $x^7-7bx^5+14b^2x^3-7b^3x+C=0$.

Notand $C=7a_4$; $-7b^3=a_3$; $14b^2=a_2$;; $-7b=a_1$ ecuatia (3) devine ecuatia f(x)=0.

Mai mult: $7a_2=7\cdot 14b^2=2a_1^2$ si $49a_3=(-7b)^3=a_1^3$.

Asadar, se confirma relatiile (2).

Solutia 2
Aceasta solutie vine in completarea solutiei 1,deoarece vom rezolva ecuatia afland efectiv forma solutiilor.

Vom cauta pentru ecuatia f(x)=0 solutii reale nenule de forma :
$x_0=\alpha+\frac{k}{\alpha}$, cu $k\in\mathbf{N}$,$k\geq 2$ si $\alpha\in\mathbf{R}-\mathbf{Q}$ **.**

Urmarim ideea ca printr-o substitutie de forma $x\to x+\frac{k}{x}$,$x\neq 0$ sa ajungem la un
polinom de forma:
(4) $g=x^7+\frac{M}{x^7}+a_4$, cu $x\neq 0$, $a_4\neq 0$. Calculam deci $g(x)=f(x+\frac{k}{x})$.

Obtinem succesiv:

$(x+\frac{k}{x})^7 = x^7+7k\,x^5+21k^2x^3+35k^3\,x+35\frac{k^4}{x}+21\frac{k^5}{x^3}+7\frac{k^6}{x^5}+(\frac{k}{x})^7$.

$(x+\frac{k}{x})^5=x^5+5kx^3+10k^2\,x+10\frac{k^3}{x}+5\frac{k^4}{x^3}+(\frac{k}{x})^5$.

$(x+\frac{k}{x})^3=x^3+3kx+3\frac{k^2}{x}+(\frac{k}{x})^3$.

Asadar: $g(x)=x^7+(7k+a_1)x^5+(21k^2+5ka_1+a_2)x^3+(35k^3+10a_1k^2+3a_2k+a_3)x+$

$$+\frac{35k^4+10a_1k^3+3a_2k^2+a_3k}{x}+\frac{21k^5+5a_1k^4+a_2k^3}{x^3}+\frac{7k^6+a_1k^5}{x^5}+(\frac{k}{x})^7+a_4$$

Polinomul g va avea forma (4) daca au loc conditiile:

$$\begin{cases} 7k+a_1=0 \\ 21k^2+5ka_1+a_2=0 \\ 35k^3+10a_1k^2+3a_2k+a_3=0 \\ 35k^4+10a_1k^3+3a_2k^2+a_3k=0 \\ 21k^5+5a_1k^4+a_2k^3=0 \\ 7k^6+a_1k^5=0 \end{cases}$$

Rezolvand acest sistem obtinem: $a_1=-7k$; $a_2=14\,k^2$; $a_3=-7k^3$ (5)

Observatia 3

Polinomul f are forma:

$f= x^7-7k\,x^5+14k^2x^3-7k^3\,x+a_4$,cu $k\in\mathbf{N}$, $k\geq 2$, $a_4\neq 0$

Eliminand k din relatiile (5) ,regasim conditiile: $7a_2=2a_1^2$,respectiv $49a_3=a_1^3$.

In aceste conditii sa rezolvam efectiv ecuatia $f(x)=0$.

Observatia 4

Deoarece $g(x)=f(x+\frac{k}{x})$, obtinem ca, daca x_0 este solutie a ecuatiei $g(x)=0$, atunci

$x_0+\frac{k}{x_0}$ este solutie a ecuatiei f(x)=0.

Avem $g=x^7+\left(\frac{k}{x}\right)^7+a_4$. Notam $x^7=t$ si obtinem :

$g(x)=0 \Leftrightarrow t^2+a_4t+k^7=0$ unde $\Delta=a_4^2-4k^7$

a) Daca $\Delta \geq 0 \Rightarrow t_1=\frac{-a_4+\sqrt{\Delta}}{2}$ si $t_2=\frac{-a_4-\sqrt{\Delta}}{2}$

Ecuatiile $x^7=t_1$; $x^7=t_2$ admit solutiile reale:

$$x_1=\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}, \quad x_2=\sqrt[7]{\frac{-a_4-\sqrt{\Delta}}{2}}$$

Asadar $g(x_1)=g(x_2)=0 \Rightarrow f\left(x_1+\frac{k}{x_1}\right)=f\left(x_2+\frac{k}{x_2}\right)=0$.

Observatia 5

$x_1+\frac{k}{x_1}=x_2+\frac{k}{x_2}$ este adevarat deoarece $x_1x_2=k$.

Deci solutia reala a ecuatiei f(x)=0 este de forma:

$$x_0=\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}+\frac{k}{\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}} \in R\text{-}Q \qquad \text{sau}$$

$$x_0=\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}+\sqrt[7]{\frac{-a_4-\sqrt{\Delta}}{2}}\ .$$

b) Daca $\Delta<0 \Rightarrow t_1=\frac{-a_4+i\sqrt{-\Delta}}{2}$; $t_2=\frac{-a_4-i\sqrt{-\Delta}}{2}$

Din relatiile lui Viete stim ca $t_1+t_2=-a_4$ si $t_1t_2=k^7$

Daca notam cu x_1 orice solutie (complexa) a ecuatiei $x^7=t_1$ respectiv cu x_2 orice solutie (complexa) a ecuatiei $x^7=t_2$ obtinem $x_1^7x_2^7=k^7 \Leftrightarrow (x_1 x_2)^7=k^7$

Daca privim aceasta relatie ca fiind o ecuatie in necunoscuta x_1x_2 ,obtinem o solutie reala si anume $x_1x_2=k\in\mathbf{N}$

Mai mult ,deoarece $\overline{t_2}=t_1 \Leftrightarrow x_2^7=\overline{x_1}^7=(\overline{x_1})^7$, putem avea $x_2=\overline{x_1}$

Atunci solutia $x_0=x_1+\frac{k}{x_1}=x_2+\frac{k}{x_2}=x_1+x_2\in\mathbf{R}$, chiar daca $x_1, x_2\in\mathbf{C}$

In continuare vom lucra in cazul $\Delta\geq 0$. Celelalte solutii ale ecuatiei $g(x)=0$,vor fi numere complexe provenind din rezolvarea ecuatiilor $x^7=t_1$ si $x^7=t_2$

Folosind scrierea sub forma trigonometrica a numerelor t_1, t_2 obtinem:

$$x'_n=\sqrt[7]{t_1}\,[\cos(\frac{2n\pi}{7})+i\ \sin(\frac{2n\pi}{7})]\ ,\ \forall\, n=\overline{1,6} \qquad (6)$$

$$x''_n=\sqrt[7]{t_2}\,[\cos(\frac{2n\pi}{7})+i\ \sin(\frac{2n\pi}{7})]\ ,\ \forall\, n=\overline{1,6} \qquad (7)$$

Observatia 6

Evident ,ne intereseaza doar solutiile complexe ale ecuatiilor $x^7=t_1$ si $x^7=t_2$

Vom nota $z_n=\cos(\frac{2n\pi}{7})+i\ \sin(\frac{2n\pi}{7})$,$\forall\ n=\overline{1,6}$

Observatia 7

$$|z_n|=1\Rightarrow \overline{z_n}=\frac{1}{z_n} \quad ,\forall n=1,6$$

Obtinem deci $g(x'_n)=g(x''_n)=0\Rightarrow f(x'_n+\frac{k}{x'_n})=f(x''_n+\frac{k}{x''_n})$,$\forall n=1,6$

Propozitia 1

$$\overline{x_n^{'}+\frac{k}{x_n^{'}}} = x_n^{''}+\frac{k}{x_n^{''}} \qquad ,\forall n=$$

Demonstratie : $x_n^{'}+\frac{k}{x_n^{'}}=\sqrt[7]{t_1}\, z_n+\frac{k}{\sqrt[7]{t_1}\, z_n}=\sqrt[7]{t_1}\, z_n+\frac{k}{\sqrt[7]{t_1}}\,\overline{z_n}$

$$x_n^{''}+\frac{k}{x_n^{''}}=\sqrt[7]{t_2}\, z_n+\frac{k}{\sqrt[7]{t_2}\, z_n}=\sqrt[7]{t_2}\, z_n+\frac{k}{\sqrt[7]{t_2}}\,\overline{z_n}$$

Deoarece $t_1t_2=k^7 \Rightarrow \ k=\sqrt[7]{t_1}\,\sqrt[7]{t_2}$

$$\text{Atunci: } \begin{cases} x_n^{'}+\frac{k}{x_n^{'}}=\sqrt[7]{t_1}\, z_n+\sqrt[7]{t_2}\,\overline{z_n} \\ x_n^{''}+\frac{k}{x_n^{''}}=\sqrt[7]{t_2}\, z_n+\sqrt[7]{t_1}\,\overline{z_n} \end{cases}$$

Deci , propozitia 1 este evident adevarata.

Propozitia 2 $\quad \{\, x_n^{'}+\frac{k}{x_n^{'}} \ , n=\overline{1,6}\,\}=\{\, x_n^{''}+\frac{k}{x_n^{''}} \ , n=\overline{1,6}\,\}$.

Demonstratie :

Elementele primei multimi sunt numere complexe si sunt conjugate doua cate doua. Analog si elementele celei de-a doua multimi. Folosind si propozitia 1, se observa imediat egalitatea celor doua multimi.

Asadar, cele 6 solutii complexe ale ecuatiei f(x)=0 au forma:

$$x_n=\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}\, z_n + \frac{k}{\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}}\,\overline{z_n} \ \text{ sau}$$

$$x_n=\sqrt[7]{\frac{-a_4+\sqrt{\Delta}}{2}}\,z_n+\sqrt[7]{\frac{-a_4-\sqrt{\Delta}}{2}}\,\overline{z_n}$$

<u>Aplicatie</u>

Sa se arate ca numarul $x=\sqrt[7]{8}\,(1+\sqrt[7]{2})$ este solutie a ecuatiei :

$x^7-14x^5+56x^3-56x-24=0$.

Sa se rezolve apoi ecuatia.

<u>Demonstratie</u>

Evident un calcul direct este destul de complicat. Aplicand cele de mai sus avem:

$k=2, a_1=-14, a_2=56, a_3=-56, a_4=-24 \Rightarrow \Delta=64$

Atunci , solutiile sunt de forma:

$x_0=\sqrt[7]{\frac{24+8}{2}}+\sqrt[7]{\frac{24-8}{2}}=\sqrt[7]{16}+\sqrt[7]{8}$ (solutie reala)

Solutiile complexe sunt de forma :

$x_n=\sqrt[7]{16}\,z_n+\sqrt[7]{8}\,\overline{z_n}$, $\forall$ $n=\overline{1,6}$, cu $z_n=\cos(\frac{2n\pi}{7})+i\ \sin(\frac{2n\pi}{7})$.

Observatia 8

Aplicatia de mai sus poate fi privita in urmatorul context mai general:
"Sa se gaseasca o ecuatie algebrica de grad 7,f(x)=0 unde f are forma (1) care sa admita solutia reala $x_0=\sqrt[7]{k^3}+\sqrt[7]{k^4}$, unde $k\in N$, $k\geq 2$ fixat arbitrar " .

Demonstratie

Folosind notatiile din aceasta nota avem , $x_1=\sqrt[7]{k^3}$, $x_2=\sqrt[7]{k^4}$ $\Rightarrow t_1=k^3$, $t_2=k^4$. Dar $t_1+t_2=-a_4 \Rightarrow a_4=-k^3-k^4$.

Ecuatia cautata va avea forma:

$x^7 - 7kx^5 + 14k^2 x^3 - 7k^3 x_4 - k^3 - k^4 = 0$, $k \in N$, $k \geq 2$.

Observatia 9

Analog se pot construi polinoame care sa aiba forma (1) si sa admita radacini reale de tipul $x_0 = \sqrt[7]{k} + \sqrt[7]{k^6}$ sau $x_0 = \sqrt[7]{k^2} + \sqrt[7]{k^5}$, unde $k \in N$, $k \geq 2$.