Câteva tipuri de ecuaţii trigonometrice

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Ecuatiile ce contin necunoscute sub semnul functiilor trigonometrice se numesc ecuatii trigonometrice. Cele mai simple ecuatii trigonometrice sunt ecuatii le de tipul

$$\sin x = a, \quad \cos x = a, \quad \mathrm{tg} x = a, \quad \mathrm{ctg} x = a, \quad a \in \mathbf{R}. \qquad (1)$$

Cum rezolvarea ecuatiilor trigonometrice se reduce la rezolvarea ecuatiilor de tipul (1) (utilizand diferite transformari), vom aminti afirmatiile de baza referitor solutiile ecuatiilor (1).

1. Ecuatia

$$\mathbf{\sin x = a, \quad a \in R,} \qquad (2)$$

pentru $|a| > 1$ solutii nu are, iar pentru $|a| \le 1$ multimea solutiilor ei se contine in formula

$$x = (-1)^n \arcsin a + \pi n, \quad n \in \mathbf{Z}, \qquad (3)$$

unde $\arcsin a \in [-[(\pi)/ 2];[(\pi)/ 2]]$ este unghiul, sinusul caruia este egal cu a, iar **Z** desemneaza multimea numerelor intregi, sau, echivalent (tinand seama de paritatea lui n), in totaliatea

$$\left[\begin{array}{l} x = \arcsin a + 2\pi k, \\ x = \pi - \arcsin a + 2\pi k, \end{array}\right. \quad k \in \mathbf{Z}. \qquad (4)$$

Obs 1. Daca in ecuatia (2) $a \in \{0;-1;1\}$ solutiile ei (3) se scriu mai simplu, si anume

$$\sin x = 0 \iff x = \pi n, \quad n \in \mathbf{Z},$$
$$\sin x = 1 \iff x = {}^{\pi}/_{2} + 2\pi n, \quad n \in \mathbf{Z},$$
$$\sin x = -1 \iff x = -{}^{\pi}/_{2} + 2\pi n, \quad n \in \mathbf{Z}.$$

Exp 1. Sa se rezolve ecuatiile

$$\text{a) } \sin x = \frac{\sqrt{3}}{2}; \quad \text{b) } \sin x = -\frac{1}{3}; \quad \text{c) } \sin x = \sqrt{11} - 2.$$

Rezolvare. a) Cum $\left|\frac{\sqrt{3}}{2}\right| \leq 1$, conform (3) solutiile ecuatiei date sunt

$$x = (-1)^n \arcsin\frac{\sqrt{3}}{2} + \pi n,\ n \in \mathbf{Z},$$

sau tinand seama ca $\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$, se obtine

$$x = (-1)^n \frac{\pi}{6} + \pi n,\ n \in \mathbf{Z}.$$

b) Similar exemplului a) se obtine $x = (-1)^n \arcsin\left(-\frac{1}{3}\right) + \pi n,\ n \in \mathbf{Z}$ sau, tinand seama arcsinus ca functia este o functie impara,

$$x = (-1)^{n+1} \arcsin\frac{1}{3} + \pi n,\ n \in \mathbf{Z}.$$

c) Cum $\sqrt{11} - 2 > 1$, rezulta ca ecuatia data nu are solutii.

2. Ecuatia

$$\mathbf{\cos}\boldsymbol{x} = \boldsymbol{a} \tag{5}$$

pentru $|a| > 1$ nu are solutii, iar pentru $|a| \leq 1$ multimea solutiilor ei se contine in formula

$$x = \pm \arccos a + 2\pi n,\quad n \in \mathbf{Z}, \tag{6}$$

unde $\arccos a \in [0;\pi]$ este unghiul, cosinusul caruia este egal cu a.

Obs 2. Daca in ecuatia (5) $a \in \{0;1;-1\}$ solutiile ei (6) se scriu mai simplu, si anume

$$\cos x = 0 \quad \Leftrightarrow \quad x = \pi/2 + \pi n, \quad n \in \mathbf{Z},$$
$$\cos x = 1 \quad \Leftrightarrow \quad x = 2\pi n, \quad n \in \mathbf{Z},$$
$$\cos x = -1 \quad \Leftrightarrow \quad x = \pi + 2\pi n, \quad n \in \mathbf{Z}.$$

Exp 2. Sa se rezolve ecuatiile:

a) $\cos x = -1/2$; b) $\cos x = 2/3$; c) $\cos x = \frac{\sqrt{3}+1}{2}$.

Rezolvare. a) Cum $\left|-\frac{1}{2}\right| \le 1,$ conform (6) solutiile ecuatiei date sunt $x = \pm\arccos\left(-\frac{1}{2}\right) + 2\pi n,\ n \in \mathbf{N},$ sau tinand seama ca $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3},$ se obtine $x = \pm\frac{2\pi}{3} + 2\pi n,\ n \in \mathbf{Z}.$

b) Similar exemplului a) se obtine $x = \pm\arccos\frac{2}{3} + 2\pi n,\ n \in \mathbf{Z}.$

c) Cum $\frac{\sqrt{3}+1}{2} > 1,$ ecuatia data nu are solutii.

3. Ecuatia

$$\mathbf{tg}x = a,\quad a \in \mathbf{R} \tag{7}$$

are solutiile

$$x = \mathrm{arctg}\,a + \pi n,\quad n \in \mathbf{Z}, \tag{8}$$

unde $\mathrm{arctg}\,a \in (-\pi/2;\pi/2)$ este unghiul, tangenta caruia este egala cu a.

4. Ecuatia

$$\mathbf{ctg}x = a,\quad a \in \mathbf{R} \tag{9}$$

are solutiile

$$x = \mathrm{arcctg}\,a + \pi n,\quad n \in \mathbf{Z}, \tag{10}$$

unde $\mathrm{arcctg}\,a \in (0;\pi)$ este unghiul, cotangenta caruia este egala cu a.

Exp 3. Sa se rezolve ecuatiile

a) $\mathrm{tg}x = 1$; b) $\mathrm{tg}x = -2$; c) $\mathrm{ctg}x = -1$; d) $\mathrm{ctg}x = 3$.

Rezolvare. a) Conform (8) solutiile ecuatiei date sunt $x = \mathrm{arctg}1 + \pi n,\ n \in \mathbf{Z}$, sau tinand seama ca $\mathrm{arctg}\,1 = \frac{\pi}{4},$ se obtine $x = \frac{\pi}{4} + \pi n,\ n \in \mathbf{Z}.$

b) Similar exemplului precedent se obtine $x = \text{arctg}(-2) + \pi n,\ \ n \in \mathbf{Z}$, sau tinand seama ca arctangenta este o functie impara, $x = -\text{arctg}2 + \pi n,\ \ n \in \mathbf{Z}$.

c) Se tine seama de (10) si se obtine

$$x = \text{arcctg}\,(-1) + \pi n,\quad n \in \mathbf{Z},$$

sau, cum $\text{arcctg}(-1) = \frac{3\pi}{4}$, $x = \frac{3\pi}{4} + \pi n,\ n \in \mathbf{Z}$.

d) Similar exemplului c) se obtine $x = \text{arcctg}3 + \pi n,\ \ n \in \mathbf{Z}$.

Obs. Ecuatiile

$$\sin f(x) = a,\qquad \cos f(x) = a,\qquad \text{tg}\, f(x) = a,\qquad \text{ctg}\, f(x) = a \tag{11}$$

prin intermediul substitutiei $f(x) = t$ se reduc la rezolvarea ecuatiilor (1).

Exp 4. Sa se rezolve ecuatiile

a) $\sin(2x - 1) = 1$; b) $\cos(x^2 + 4) = -1$; c) $\text{tg}\, 2x = \sqrt{3}$; d) $\text{ctg}x^3 = -2$.

Rezolvare. a)

$$\sin(2x - 1) = 1 \ \Leftrightarrow\ \begin{cases} \sin t = 1, \\ t = 2x - 1, \end{cases} \ \Leftrightarrow\ 2x - 1 = \pi/2 + 2\pi n,\ \ n \in \mathbf{Z} \ \Leftrightarrow$$

$$\Leftrightarrow\ 2x = \pi/2 + 2\pi n + 1,\ \ n \in \mathbf{Z} \ \Leftrightarrow\ x = \pi/4 + \pi n + 1/2,\ \ n \in \mathbf{Z}.$$

b)

$$\cos(x^2 + 4) = -1 \ \Leftrightarrow\ \begin{cases} \cos t = -1, \\ t = x^2 + 4, \end{cases} \ \Leftrightarrow\ \begin{cases} x^2 + 4 = \pi + 2\pi n,\ \ n \in \mathbf{Z}, \\ \pi + 2\pi n \geq 4, \end{cases}$$

$$\Leftrightarrow\ x^2 = \pi + 2\pi n - 4,\ \ n = 1,2,3,... \ \Leftrightarrow\ x = \pm\sqrt{\pi + 2\pi n - 4},\ n = 1,2,3,...$$

(se tine seama ca radicalul de ordin par exista doar din valori nenegative).

c) $\text{tg}\, 2x = \sqrt{3} \ \Leftrightarrow\ 2x = \text{arctg}\sqrt{3} + \pi n,\ n \in \mathbf{Z} \ \Leftrightarrow\ 2x = \pi/3 + \pi n,\ \ n \in \mathbf{Z} \ \Leftrightarrow$ $x = \frac{\pi}{6} + \frac{\pi}{2} n,\quad n \in \mathbf{Z}$.

d) $\text{ctg}x^3 = -2 \ \Leftrightarrow\ x^3 = \text{arcctg}(-2) + \pi n,\ \ n \in \mathbf{Z} \ \Leftrightarrow\ x = \sqrt[3]{\text{arcctg}(-2) + \pi n},\ n \in \mathbf{Z}$

Ecuatii trigonometrice reductibile la ecuatii de gradul al doilea

Ecuatia

$$\boldsymbol{a\sin^2 x + b\sin x + c = 0, \quad a, b, c \in \mathbf{R}, \quad a \neq 0} \tag{12}$$

prin intermediul substitutiei $t = \sin x$, $(|t| \leq 1)$ se reduce la ecuatia patrata $at^2 + bt + c = 0$.

Exp 5. Sa se rezolve ecuatiile

a) $2\sin^2 x - 5\sin x + 2 = 0$; b) $\sin^2 2x - \sin 2x = 0$; c) $\sin^2 x - \sin x + 6 = 0$.

Rezolvare. a) Se noteaza $\sin x = t$ si ecuatia devine

$$2t^2 - 5t + 2 = 0,$$

de unde $t_1 = {}^1/_2$ si $t_2 = 2$. Cum $|t| \leq 1$, ramane $t = {}^1/_2$ si prin urmare ecuatia initiala este echivalenta cu ecuatia

$$\sin x = {}^1/_2,$$

solutiile careia sunt (a se vedea (3)) $x = (-1)^n \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}.$

b) Se noteaza $\sin x = t$ si se obtine ecuatia patrata $t^2 - t = 0$ cu solutiile $t_1 = 0$ si $t_2 = 1$. Astfel ecuatia initiala este echivalenta cu totalitatea de ecuatii

$$\left[\begin{array}{l} \sin 2x = 0, \\ \sin 2x = 1, \end{array}\right.$$

de unde

$$\left[\begin{array}{l} x = \frac{\pi}{2} n, \quad n \in \mathbf{Z}, \\ x = \frac{\pi}{4} + \pi k, \quad k \in \mathbf{Z}. \end{array}\right.$$

c) Similar exemplelor precedente se obtine ecuatia patrata $t^2 - t + 6 = 0$, care nu are solutii. Rezulta ca si ecuatia trigonometrica nu are solutii.

Ecuatiile

$$a\cos^2 x + b\cos x + c = 0, \tag{13}$$

$$a\,\mathrm{tg}^2 x + b\,\mathrm{tg} x + c = 0, \tag{14}$$

$$a\,\mathrm{ctg}^2 x + b\,\mathrm{ctg} x + c = 0, \tag{15}$$

unde $a, b, c \in \mathbf{R}$, $a \neq 0$ se rezolva similar ecuatiei (12).

In cazul ecuatiei (13) se tine seama ca t = cos$x$ in modul urmeaza sa nu intreaca unu, iar pentru t = tg$x$ (t = ctg$x$) in ecuatia (14) (respectiv (15)) restrictii nu sunt.

Exp 6. Sa se rezolve ecuatiile

a) $6\cos^2 x - 5\cos x + 1 = 0$; b) $\mathrm{tg}^2 2x - 4\mathrm{tg}2x + 3 = 0$; c) $\mathrm{ctg}^2\frac{x}{2} - \mathrm{ctg}\frac{x}{2} - 2 = 0$.

Rezolvare. a) Se noteaza cos$x$ = t si se obtine ecuatia patrata

$$6t^2 - 5t + 1 = 0$$

cu solutiile $t = {}^1/_3$ si $t_2 = {}^1/_2$. Cum ambele solutii verifica conditia $|t| \le 1$ se obtine totalitatea

$$\left[\begin{array}{l} \cos x = {}^1/_3, \\ \cos x = {}^1/_2, \end{array}\right.$$

de unde $x = \pm\arccos\frac{1}{3} + 2\pi n,\ n \in \mathbf{Z},\quad x = \pm\frac{\pi}{3} + 2\pi k,\ k \in \mathbf{Z}.$

b) Se noteaza tg$2x$ = t si se obtine ecuatia patrata

$$t^2 - 4t + 3 = 0$$

cu solutiile $t_1 = 1$ si $t_2 = 3$. Prin urmare

$$\left[\begin{array}{l} \mathrm{tg}2x = 1, \\ \mathrm{tg}2x = 3, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 2x = \frac{\pi}{4} + \pi n,\ n \in \mathbf{Z}, \\ 2x = \mathrm{arctg}3 + \pi k,\quad k \in \mathbf{Z}, \end{array}\right.$$

de unde $x = \frac{\pi}{8} + \frac{\pi}{2}n,\quad x = \frac{1}{2}\mathrm{arctg}\,3 + \frac{\pi}{2}k,\ n, k \in \mathbf{Z}.$

c) Se rezolva similar exemplului precedent si se obtine $x = \frac{3\pi}{2} + 2\pi n,$ $x = 2\mathrm{arcctg}2 + 2\pi k,\quad n, k \in \mathbf{Z}.$

Ecuatia

$$a\cos^2 x + b\sin x + c = 0, \tag{16}$$

utilizand identitatea trigonometrica de baza $\sin^2 x + \cos^2 x = 1$, se reduce la rezolvarea unei ecuatii de tipul (12):

$$a(1 - \sin^2 x) + b\sin x + c = 0.$$

Similar, ecuatia

$$a\sin^2 x + b\cos x + c = 0 \tag{17}$$

se reduce la rezolvarea unei ecuatii de tipul (13):

$$a(1 - \cos^2 x) + b\cos x + c = 0.$$

Utilizand formulele

$$\cos 2x = 1 - 2\sin^2 x, \quad \cos 2x = 2\cos^2 x - 1$$

ecuatiile

$$a\cos 2x + b\sin x + c = 0, \tag{18}$$

$$a\cos 2x + b\cos x + c = 0, \tag{19}$$

se reduc la rezolvarea ecuatiilor de tipul (12) si respectiv (13).

Bibliografie:

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