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ARTICOLE REVISTĂ:

1. **Obținerea unor relații în triunghi cu ajutorul inegalității lui Radon … pag. 2**

Marin Chirciu

2. **A Result by Using Beta Function … pag. 41**

Toyesh Prakash Sharma

3. **Trei soluţii ale unei probleme din RMT2/2020 ... pag. 44**

Daniel Văcaru

4. **Teorema chinezească a resturilor și exemple... pag. 45**

Petreanu Irina

5. **PROBLEME INTERESANTE... pag. 48**

Anton Flavia

1. Obținerea unor relații în triunghi cu ajutorul inegalității lui Radon

Marin Chirciu[1]
Art 2403

Articolul propune folosirea inegalității lui Radon pentru obținerea unor relații în triunghi.

Radon-Inequality

If $x_k, y_k > 0, \forall k = \overline{1,n}$, $n \geq 2$ and $t \geq 0$, then

$$\frac{x_1^{t+1}}{y_1^t} + \frac{x_2^{t+1}}{y_2^t} + \ldots + \frac{x_n^{t+1}}{y_n^t} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots + y_n)^t}.$$

Equality holds if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

Vom folosi inegalitatea lui Radon în cazul $n = 3$.

Inegalitatea lui Radon

If $x, y, z, a, b, c > 0$ and $n \geq 0$ then

$$\frac{x^{n+1}}{a^n} + \frac{y^{n+1}}{b^n} + \frac{z^{n+1}}{c^n} \geq \frac{(x+y+z)^{n+1}}{(a+b+c)^n}.$$

Equality holds if and only if $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Să trecem la aplicații în triunghi.

Aplicații

Aplicația 1

In ΔABC

$$\sum \sqrt{a(b+c-a)} \geq 6r\sqrt{3}.$$

Soluție.

Folosim inegalitatea lui **Radon** obținem:

$$Ms = \sum \sqrt{a(b+c-a)} = \sum \frac{a^{\frac{1}{2}}}{(b+c-a)^{\frac{-1}{2}}} \overset{Radon}{\geq} \frac{\left(\sum a\right)^{\frac{1}{2}}}{\left[\sum (b+c-a)\right]^{\frac{-1}{2}}} = \frac{(2p)^{\frac{3}{2}}}{(2p)^{\frac{1}{2}}} = 2p \overset{Mitrinovic}{\geq} 2 \cdot 3r\sqrt{3} =$$

$$= 6r\sqrt{3} = Md.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația 2

In ΔABC

$$\sum a\sqrt{\frac{a}{b+c-a}} \geq 6r\sqrt{3}.$$

Soluție.

$$Ms = \sum a\sqrt{\frac{a}{b+c-a}} = \sum \frac{a^{\frac{3}{2}}}{(b+c-a)^{\frac{1}{2}}} \overset{Radon}{\geq} \frac{\left(\sum a\right)^{\frac{3}{2}}}{\left[\sum (b+c-a)\right]^{\frac{1}{2}}} = \frac{(2p)^{\frac{3}{2}}}{(2p)^{\frac{1}{2}}} = 2p \overset{Mitrinovic}{\geq} 2 \cdot 3r\sqrt{3} =$$

$$= 6r\sqrt{3} = Md.$$

[1] *Profesor, Colegiul Național „Zinca Golescu”, Pitești*

Aplicația 3

In ΔABC

$$\sum\left(\frac{a}{b+c-a}\right)^n\sqrt{a(b+c-a)}\geq 6r\sqrt{3}\,,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\left(\frac{a}{b+c-a}\right)^n\sqrt{a(b+c-a)}=\sum\frac{a^{n+\frac{1}{2}}}{(b+c-a)^{n-\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum a\right)^{n+\frac{1}{2}}}{\left[\sum(b+c-a)\right]^{n-\frac{1}{2}}}=\frac{(2p)^{n+\frac{1}{2}}}{(2p)^{n-\frac{1}{2}}}=$$

$$=2p\overset{Mitrinovic}{\geq}2\cdot 3r\sqrt{3}==6r\sqrt{3}=Md.$$

Aplicația 4

In acute ΔABC

$$\sum\frac{a\sqrt{a}}{\sqrt{b^2+c^2-a^2}}\geq\frac{2r}{R}\sqrt{6p}.$$

Soluție.

$$Ms=\sum\frac{a^{\frac{3}{2}}}{\left(b^2+c^2-a^2\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum a\right)^{\frac{3}{2}}}{\left[\sum\left(b^2+c^2-a^2\right)\right]^{\frac{1}{2}}}=\frac{(2p)^{\frac{3}{2}}}{\left(\sum a^2\right)^{\frac{1}{2}}}=\frac{2p\sqrt{2p}}{\sqrt{\sum a^2}}\overset{Leibniz}{\geq}\frac{2p\sqrt{2p}}{\sqrt{9R^2}}=$$

$$=\frac{2p\sqrt{2p}}{3R}\overset{Mitrinovic}{\geq}\frac{2\cdot 3r\sqrt{3}\cdot\sqrt{2p}}{3R}=\frac{2r}{R}\sqrt{6p}=Md.$$

Aplicația 5

In acute ΔABC

$$\sum\sqrt{a\left(b^2+c^2-a^2\right)}\geq 6r\sqrt{2p}.$$

Soluție.

$$Ms=\sum\sqrt{a\left(b^2+c^2-a^2\right)}=\sum\frac{a^{\frac{1}{2}}}{\left(b^2+c^2-a^2\right)^{\frac{-1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum a\right)^{\frac{1}{2}}}{\left[\sum\left(b^2+c^2-a^2\right)\right]^{\frac{-1}{2}}}=\frac{(2p)^{\frac{1}{2}}}{\left(\sum a^2\right)^{\frac{-1}{2}}}=$$

$$=\sqrt{2p}\sqrt{\sum a^2}\overset{Neuberg}{\geq}\sqrt{2p}\sqrt{36r^2}=6r\sqrt{2p}.$$

Aplicația 6

In acute ΔABC

$$\sum\frac{a^2}{b^2+c^2-a^2}\geq\left(\frac{2p}{3R}\right)^2.$$

Soluție.

Aplicația 7

In acute ΔABC

$$\sum\left(\frac{a}{b^2+c^2-a^2}\right)^n\sqrt{a\left(b^2+c^2-a^2\right)}\geq\left(\frac{2p}{9R^2}\right)^n 3R\sqrt{2p}\,,n\in\mathbf{N}^*.$$

Marin Chirciu

Soluție.

$$Ms=\sum\left(\frac{a}{b^2+c^2-a^2}\right)^n\sqrt{a\left(b^2+c^2-a^2\right)}\sum\frac{a^{n+\frac{1}{2}}}{\left(b^2+c^2-a^2\right)^{n-\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum a\right)^{n+\frac{1}{2}}}{\left[\sum\left(b^2+c^2-a^2\right)\right]^{n-\frac{1}{2}}}=$$

$$=\frac{\left(2p\right)^{n+\frac{1}{2}}}{\left(\sum a^2\right)^{n-\frac{1}{2}}}\overset{Leibniz}{\geq}\frac{\left(2p\right)^{n+\frac{1}{2}}}{\left(9R^2\right)^{n-\frac{1}{2}}}=\left(\frac{2p}{9R^2}\right)^n 3R\sqrt{2p}=Md\ .$$

Aplicația 8

In ΔABC

$$\sum\sqrt{m_a\left(m_b+m_c-m_a\right)}\geq 9r\ .$$

Soluție.

$$Ms=\sum\sqrt{m_a\left(m_b+m_c-m_a\right)}=\sum\frac{\left(m_a\right)^{\frac{1}{2}}}{\left(m_b+m_c-m_a\right)^{\frac{-1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum m_a\right)^{\frac{1}{2}}}{\left[\sum m_a\right]^{\frac{-1}{2}}}=\sum m_a\geq 9r=Md\ .$$

Aplicația 9

In ΔABC,

$$\sum m_a\sqrt{\frac{m_a}{m_b+m_c-m_a}}\geq 9r\ .$$

Soluție.

$$Ms=\sum m_a\sqrt{\frac{m_a}{m_b+m_c-m_a}}=\sum\frac{\left(m_a\right)^{\frac{3}{2}}}{\left(m_b+m_c-m_a\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum m_a\right)^{\frac{3}{2}}}{\left[\sum\left(m_b+m_c-m_a\right)\right]^{\frac{1}{2}}}=\frac{\left(\sum m_a\right)^{\frac{3}{2}}}{\left(\sum m_a\right)^{\frac{1}{2}}}=$$

$$=\sum m_a\geq 9r=Md\ .$$

Aplicația 10

In ΔABC

$$\sum\left(\frac{m_a}{m_b+m_c-m_a}\right)^n\sqrt{m_a\left(m_b+m_c-m_a\right)}\geq 9r\ ,n\in\mathbf{N}\ .$$

Marin Chirciu

Soluție.

Obținem:

$$Ms=\sum\left(\frac{m_a}{m_b+m_c-m_a}\right)^n\sqrt{m_a\left(m_b+m_c-m_a\right)}=\sum\frac{\left(m_a\right)^{n+\frac{1}{2}}}{\left(m_b+m_c-m_a\right)^{n-\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum m_a\right)^{n+\frac{1}{2}}}{\left[\sum\left(m_b+m_c-m_a\right)\right]^{n-\frac{1}{2}}}=$$

$$=\frac{\left(\sum m_a\right)^{n+\frac{1}{2}}}{\left(\sum m_a\right)^{n-\frac{1}{2}}}=\sum m_a\geq 9r=Md\ .$$

Aplicația 11

In ΔABC, $\left(h_a,h_b,h_c\right)$ pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\sqrt{h_a\left(h_b+h_c-h_a\right)}\geq 9r\ .$$

Soluție.

$$Ms=\sum\sqrt{h_a\left(h_b+h_c-h_a\right)}=\sum\frac{\left(h_a\right)^{\frac{1}{2}}}{\left(h_b+h_c-h_a\right)^{\frac{-1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum h_a\right)^{\frac{1}{2}}}{\left[\sum h_a\right]^{\frac{-1}{2}}}=\sum h_a\geq 9r=Md\,.$$

Aplicația 12

In $\Delta ABC,\left(h_a,h_b,h_c\right)$ pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum h_a\sqrt{\frac{h_a}{h_b+h_c-h_a}}\geq 9r\,.$$

Soluție.

$$Ms=\sum h_a\sqrt{\frac{h_a}{h_b+h_c-h_a}}=\sum\frac{\left(h_a\right)^{\frac{3}{2}}}{\left(h_b+h_c-h_a\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum h_a\right)^{\frac{3}{2}}}{\left[\sum\left(h_b+h_c-h_a\right)\right]^{\frac{1}{2}}}=\frac{\left(\sum h_a\right)^{\frac{3}{2}}}{\left(\sum h_a\right)^{\frac{1}{2}}}=$$

$$=\sum h_a\geq 9r=Md\,.$$

Aplicația 13

In $\Delta ABC,\left(h_a,h_b,h_c\right)$ pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\left(\frac{h_a}{h_b+h_c-h_a}\right)^n\sqrt{h_a\left(h_b+h_c-h_a\right)}\geq 9r\,,n\in\mathbf{N}\,.$$

Marin Chirciu

Soluție.

$$Ms=\sum\left(\frac{h_a}{h_b+h_c-h_a}\right)^n\sqrt{h_a\left(h_b+h_c-h_a\right)}=\sum\frac{\left(h_a\right)^{n+\frac{1}{2}}}{\left(h_b+h_c-h_a\right)^{n-\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum h_a\right)^{n+\frac{1}{2}}}{\left[\sum\left(h_b+h_c-h_a\right)\right]^{n-\frac{1}{2}}}=$$

$$=\frac{\left(\sum h_a\right)^{n+\frac{1}{2}}}{\left(\sum h_a\right)^{n-\frac{1}{2}}}=\sum h_a\geq 9r=Md\,.$$

Aplicația 14

In $\Delta ABC,\left(w_a,w_b,w_c\right)$ pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\sqrt{w_a\left(w_b+w_c-w_a\right)}\geq 9r\,.$$

Soluție.

$$Ms=\sum\sqrt{w_a\left(w_b+w_c-w_a\right)}=\sum\frac{\left(w_a\right)^{\frac{1}{2}}}{\left(w_b+w_c-w_a\right)^{\frac{-1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum w_a\right)^{\frac{1}{2}}}{\left[\sum w_a\right]^{\frac{-1}{2}}}=\sum w_a\geq 9r=Md\,.$$

Aplicația 15

In $\Delta ABC,\left(w_a,w_b,w_c\right)$ pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum w_a\sqrt{\frac{w_a}{w_b+w_c-w_a}}\geq 9r\,.$$

Soluție.

$$Ms=\sum w_a\sqrt{\frac{w_a}{w_b+w_c-w_a}}=\sum\frac{\left(w_a\right)^{\frac{3}{2}}}{\left(w_b+w_c-w_a\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum w_a\right)^{\frac{3}{2}}}{\left[\sum\left(w_b+w_c-w_a\right)\right]^{\frac{1}{2}}}=\frac{\left(\sum w_a\right)^{\frac{3}{2}}}{\left(\sum w_a\right)^{\frac{1}{2}}}=$$

$$=\sum w_a\geq 9r=Md\,.$$

Aplicația 16

In ΔABC, (w_a, w_b, w_c) pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\left(\frac{w_a}{w_b+w_c-w_a}\right)^n\sqrt{w_a\left(w_b+w_c-w_a\right)}\ge 9r,\, n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\left(\frac{w_a}{w_b+w_c-w_a}\right)^n\sqrt{w_a\left(w_b+w_c-w_a\right)}=\sum\frac{\left(w_a\right)^{n+\frac{1}{2}}}{\left(w_b+w_c-w_a\right)^{n-\frac{1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum w_a\right)^{n+\frac{1}{2}}}{\left[\sum\left(w_b+w_c-w_a\right)\right]^{n-\frac{1}{2}}}=$$

$$=\frac{\left(\sum w_a\right)^{n+\frac{1}{2}}}{\left(\sum w_a\right)^{n-\frac{1}{2}}}=\sum w_a\ge 9r=Md.$$

Aplicația 17

In ΔABC, (r_a, r_b, r_c) pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\sqrt{r_a\left(r_b+r_c-r_a\right)}\ge 9r.$$

Soluție.

$$Ms=\sum\sqrt{r_a\left(r_b+r_c-w_a\right)}=\sum\frac{\left(r_a\right)^{\frac{1}{2}}}{\left(r_b+r_c-r_a\right)^{\frac{-1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum r_a\right)^{\frac{1}{2}}}{\left[\sum r_a\right]^{\frac{-1}{2}}}=\sum r_a\ge 9r=Md.$$

Aplicația 18

In ΔABC, (r_a, r_b, r_c) pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum r_a\sqrt{\frac{r_a}{r_b+r_c-r_a}}\ge 9r.$$

Soluție.

$$Ms=\sum r_a\sqrt{\frac{r_a}{r_b+r_c-r_a}}=\sum\frac{\left(r_a\right)^{\frac{3}{2}}}{\left(r_b+r_c-r_a\right)^{\frac{1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum r_a\right)^{\frac{3}{2}}}{\left[\sum\left(r_b+r_c-r_a\right)\right]^{\frac{1}{2}}}=\frac{\left(\sum r_a\right)^{\frac{3}{2}}}{\left(\sum r_a\right)^{\frac{1}{2}}}=$$

$$=\sum r_a\ge 9r=Md.$$

Aplicația 19

In ΔABC, (r_a, r_b, r_c) pot reprezenta lungimile laturilor unui triunghi. Atunci:

$$\sum\left(\frac{r_a}{r_b+r_c-r_a}\right)^n\sqrt{r_a\left(r_b+r_c-r_a\right)}\ge 9r,\, n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\left(\frac{r_a}{r_b+r_c-r_a}\right)^n\sqrt{r_a\left(r_b+r_c-{}_a\right)}=\sum\frac{\left(r_a\right)^{n+\frac{1}{2}}}{\left(r_b+r_c-r_a\right)^{n-\frac{1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum r_a\right)^{n+\frac{1}{2}}}{\left[\sum\left(r_b+r_c-r_a\right)\right]^{n-\frac{1}{2}}}=$$

$$=\frac{\left(\sum r_a\right)^{n+\frac{1}{2}}}{\left(\sum r_a\right)^{n-\frac{1}{2}}}=\sum r_a\ge 9r=Md.$$

Aplicația 20

In ΔABC

$$\sqrt[3]{\frac{a^4}{b^2+c(a+b)}}+\sqrt[3]{\frac{b^4}{c^2+a(b+c)}}+\sqrt[3]{\frac{c^4}{a^2+b(c+a)}}\ge 3\sqrt[3]{4r^2}.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=a, y=b, z=c$ este suficient să arătăm că: $\left(\sum a\right)^{\frac{2}{3}}\ge 3\sqrt[3]{4r^2}$,

care rezultă din: $\left(\sum a\right)^{\frac{2}{3}}\ge 3\sqrt[3]{4r^2}\Leftrightarrow(2p)^{\frac{2}{3}}\ge 3\sqrt[3]{4r^2}$,

adevărată din inegalitatea lui Mitrinovic: $p\ge 3r\sqrt{3}$. Rămâne să arătăm că:

$\left(2\cdot 3r\sqrt{3}\right)^{\frac{2}{3}}\ge 3\sqrt[3]{4r^2}\Leftrightarrow\left(2\cdot 3r\sqrt{3}\right)^{\frac{2}{3}}\ge 3\sqrt[3]{4r^2}\Leftrightarrow\left(6r\sqrt{3}\right)^2\ge 27\cdot 4r^2\Leftrightarrow 108r^2\ge 108r^2$, evident.

Aplicația 21

In ΔABC

$$\sqrt[3]{\frac{h_a^4}{h_b^2+h_c(h_a+h_b)}}+\sqrt[3]{\frac{h_b^4}{h_c^2+h_a(h_b+h_c)}}+\sqrt[3]{\frac{h_c^4}{h_a^2+h_b(h_c+h_a)}}\ge 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=h_a, y=h_b, z=h_c$ este suficient să arătăm că: $\left(\sum h_a\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2}$,
care rezultă din: $\sum h_a \geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2} \Leftrightarrow (9r)^2 \geq 27\cdot 3r^2 \Leftrightarrow 81r^2 \geq 81r^2$, evident.

Aplicația 22

In ΔABC

$$\sqrt[3]{\frac{r_a^4}{r_b^2+r_c\left(r_a+r_b\right)}}+\sqrt[3]{\frac{r_b^4}{r_c^2+r_a\left(r_b+r_c\right)}}+\sqrt[3]{\frac{r_c^4}{r_a^2+r_b\left(r_c+r_a\right)}} \geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=r_a, y=r_b, z=r_c$ este suficient să arătăm că: $\left(\sum r_a\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2}$,
care rezultă din: $\sum r_a \geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2} \Leftrightarrow (9r)^2 \geq 27\cdot 3r^2 \Leftrightarrow 81r^2 \geq 81r^2$, evident.

Aplicația 23

In ΔABC

$$\sqrt[3]{\frac{m_a^4}{m_b^2+m_c\left(m_a+m_b\right)}}+\sqrt[3]{\frac{m_b^4}{m_c^2+m_a\left(m_b+m_c\right)}}+\sqrt[3]{\frac{m_c^4}{m_a^2+m_b\left(m_c+m_a\right)}} \geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=m_a, y=m_b, z=m_c$ este suficient să arătăm că: $\left(\sum m_a\right)^{\frac{2}{3}}\ge 3\sqrt[3]{3r^2}$, care rezultă din: $\sum m_a\ge 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\ge 3\sqrt[3]{3r^2}\Leftrightarrow (9r)^2\ge 27\cdot 3r^2\Leftrightarrow 81r^2\ge 81r^2$, evident.

Aplicația 24

In ΔABC

$$\sqrt[3]{\frac{w_a^4}{w_b^2+w_c\left(w_a+w_b\right)}}+\sqrt[3]{\frac{w_b^4}{w_c^2+w_a\left(w_b+w_c\right)}}+\sqrt[3]{\frac{m_c^4}{w_a^2+w_b\left(w_c+w_a\right)}}\ge 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=w_a, y=w_b, z=w_c$ este suficient să arătăm că: $\left(\sum w_a\right)^{\frac{2}{3}}\ge 3\sqrt[3]{3r^2}$, care rezultă din: $\sum w_a\ge 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\ge 3\sqrt[3]{3r^2}\Leftrightarrow (9r)^2\ge 27\cdot 3r^2\Leftrightarrow 81r^2\ge 81r^2$, evident.

Aplicația 25

In ΔABC

$$\sqrt[3]{\frac{s_a^4}{s_b^2+s_c\left(s_a+s_b\right)}}+\sqrt[3]{\frac{s_b^4}{s_c^2+s_a\left(s_b+s_c\right)}}+\sqrt[3]{\frac{s_c^4}{s_a^2+s_b\left(s_c+s_a\right)}}\ge 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=s_a, y=s_b, z=s_c$ este suficient să arătăm că: $\left(\sum s_a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}$, care rezultă din: $\sum s_a\geq\sum h_a\geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}\Leftrightarrow(9r)^2\geq 27\cdot 3r^2\Leftrightarrow 81r^2\geq 81r^2$, evident.

Aplicația 26

In ΔABC

$$\sum\sqrt[3]{\frac{\tan^4\frac{A}{2}}{\tan^2\frac{B}{2}+\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)}}\geq\sqrt[3]{3}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\tan\frac{A}{2}\right)^{\frac{2}{3}}\geq\sqrt[3]{3}$, care rezultă din: $\left(\sum\tan\frac{A}{2}\right)^{\frac{2}{3}}=\left(\frac{4R+r}{p}\right)^{\frac{2}{3}}\overset{Doucet}{\geq}\left(\sqrt{3}\right)^{\frac{2}{3}}=\sqrt[3]{3}$.

Aplicația 27

In ΔABC

$$\sum\sqrt[3]{\frac{\cot^4\frac{A}{2}}{\cot^2\frac{B}{2}+\cot\frac{C}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)}}\geq\sqrt[3]{3}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\cot\frac{A}{2}, y=\cot\frac{B}{2}, z=\cot\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\cot\frac{A}{2}\right)^{\frac{2}{3}}\geq 3$, care rezultă din: $\left(\sum\cot\frac{A}{2}\right)^{\frac{2}{3}}=\left(\frac{p}{3}\right)^{\frac{2}{3}}\overset{Mitrinovic}{\geq}\left(3\sqrt{3}\right)^{\frac{2}{3}}=\sqrt[3]{27}=3$.

Aplicația 28

In ΔABC

$$\sum\sqrt[3]{\frac{\sin^4\frac{A}{2}}{\sin^2\frac{B}{2}+\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}}\geq\left(1+\frac{r}{R}\right)^{\frac{2}{3}}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, z=\sin\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\sin\frac{A}{2}\right)^{\frac{2}{3}}\geq\left(1+\frac{r}{R}\right)^{\frac{2}{3}}$, care rezultă din: $\sum\sin\frac{A}{2}\geq 1+\frac{r}{R}$, adevărată din

$$1+\frac{r}{R}=\sum\cos A=\sum\cos\frac{B+C}{2}\cos\frac{B-C}{2}\le\sum\cos\frac{B+C}{2}=\sum\sin\frac{A}{2}.$$

Aplicația 29

In ΔABC

$$\sum\sqrt[3]{\frac{\cos^4\frac{A}{2}}{\cos^2\frac{B}{2}+\cos\frac{C}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)}}\ge 3\left(\frac{1}{4}+\frac{r}{2R}\right)^{\frac{2}{3}}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\cos\frac{A}{2},y=\cos\frac{B}{2},z=\cos\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\cos\frac{A}{2}\right)^{\frac{2}{3}}\ge 3\left(\frac{1}{4}+\frac{r}{2R}\right)^{\frac{2}{3}}$, care rezultă din: $\sum\cos\frac{A}{2}\ge 3\sqrt{3}\left(\frac{1}{2}+\frac{r}{R}\right)$, adevărată din:

Se aplică inegalitatea lui Popoviciu funcției concave $f:(0,\pi)\to R,\ f(x)=\sin x$.

Avem: $\frac{\sin A+\sin B+\sin C}{3}+\sin\frac{A+B+C}{3}\le\frac{2}{3}\sum\sin\frac{B+C}{2}\Leftrightarrow\frac{1}{3}\cdot\frac{p}{R}+\frac{\sqrt{3}}{2}\le\frac{2}{3}\sum\cos\frac{A}{2}\Leftrightarrow$

$\Leftrightarrow\sum\cos\frac{A}{2}\ge\frac{p}{2R}+\frac{3\sqrt{3}}{4}.$

Apoi: $\sum\cos\frac{A}{2}\ge\frac{p}{2R}+\frac{3\sqrt{3}}{4}\overset{Mitrinovic}{\ge}\frac{3r\sqrt{3}}{2R}+\frac{3\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}\left(\frac{r}{R}+\frac{1}{2}\right).$

Aplicația 30

In ΔABC

$$\sum\sqrt[3]{\frac{\sec^4\frac{A}{2}}{\sec^2\frac{B}{2}+\sec\frac{C}{2}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)}}\ge\sqrt[3]{12}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\sec\frac{A}{2}, y=\sec\frac{B}{2}, z=\sec\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\sec\frac{A}{2}\right)^{\frac{2}{3}}\ge\left(2\sqrt{3}\right)^{\frac{2}{3}}=\sqrt[3]{12}$, care rezultă din: $\sum\sec\frac{A}{2}\ge2\sqrt{3}$, adevărată din:

$$\sum\sec\frac{A}{2}=\sum\frac{1}{\cos\frac{A}{2}}\overset{AHM}{\ge}\frac{9}{\sum\cos\frac{A}{2}}\overset{Jensen}{\ge}\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}.$$

Aplicația 31

In ΔABC

$$\sum\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}}\ge\sqrt[3]{36}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\ge\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\csc\frac{A}{2}, y=\csc\frac{B}{2}, z=\csc\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\csc\frac{A}{2}\right)^{\frac{2}{3}}\ge(6)^{\frac{2}{3}}=\sqrt[3]{36}$, care rezultă din: $\sum\csc\frac{A}{2}\ge6$, adevărată din:

$$\sum \csc\frac{A}{2}=\sum\frac{1}{\sin\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\sin\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3}{2}}=6.$$

Aplicația 32

If $x,y,z>0$ such that $xy+yz+zx=3$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq 1.$$

Kostas Geronikolas, Greece

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** este suficient să arătăm că: $\frac{1}{3}\sum x\geq 1$, care rezultă din:

$$\left(\sum x\right)^2\geq 3\sum yz=3\cdot 3=9.$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{\sqrt{3}}$.

Aplicația 33

If $x,y,z>0$ such that $xy+yz+zx=3$ and $\lambda\geq 0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{3}{\sqrt{\lambda+1}}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** este suficient să arătăm că: $\frac{1}{\sqrt{\lambda+1}}\sum x\geq\frac{3}{\sqrt{\lambda+1}}$, care rezultă din:

$$\left(\sum x\right)^2\geq 3\sum yz=3\cdot 3=9.$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{\sqrt{3}}$.

Aplicația 34

In ΔABC

$$\sqrt{\frac{a^3}{a+8b}}+\sqrt{\frac{b^3}{b+8c}}+\sqrt{\frac{c^3}{c+8a}}\geq\frac{2p}{3}.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=a,y=b,z=c$ obținem:

$$Ms=\sqrt{\frac{a^3}{a+8b}}+\sqrt{\frac{b^3}{b+8c}}+\sqrt{\frac{c^3}{c+8a}}\overset{Lema}{\geq}\frac{1}{3}\sum a=\frac{2p}{3}=Md.$$

Aplicația 35

In ΔABC

$$\sqrt{\frac{a^3}{a+\lambda b}}+\sqrt{\frac{b^3}{b+\lambda c}}+\sqrt{\frac{c^3}{c+\lambda a}}\geq\frac{2p}{\sqrt{\lambda+1}},\text{ unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=a,y=b,z=c$ obținem:

$$\sqrt{\frac{a^3}{a+\lambda b}}+\sqrt{\frac{b^3}{b+\lambda c}}+\sqrt{\frac{c^3}{c+\lambda a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum a=\frac{2p}{\sqrt{\lambda+1}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral sau $\lambda=0$.

Aplicația 36

In ΔABC

$$\sqrt{\frac{a^3}{a+8b}}+\sqrt{\frac{b^3}{b+8c}}+\sqrt{\frac{c^3}{c+8a}}\geq\frac{2p}{3}.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=a, y=b, z=c$ obținem:

$$Ms=\sqrt{\frac{a^3}{a+8b}}+\sqrt{\frac{b^3}{b+8c}}+\sqrt{\frac{c^3}{c+8a}}\overset{Lema}{\geq}\frac{1}{3}\sum a=\frac{2p}{3}=Md.$$

Aplicația 37

In ΔABC

$$\sqrt{\frac{h_a^3}{h_a+8h_b}}+\sqrt{\frac{h_b^3}{h_b+8h_c}}+\sqrt{\frac{h_c^3}{h_c+8h_a}}\geq 3r.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=h_a, y=h_b, z=h_c$ obținem:

$$\sqrt{\frac{h_a^3}{h_a+8h_b}}+\sqrt{\frac{h_b^3}{h_b+8h_c}}+\sqrt{\frac{h_c^3}{h_c+8h_a}}\overset{Lema}{\geq}\frac{1}{3}\sum h_a\geq\frac{1}{3}\cdot 9r=3r.$$

Aplicația 38

In ΔABC

$$\sqrt{\frac{h_a^3}{h_a+\lambda h_b}}+\sqrt{\frac{h_b^3}{h_b+\lambda h_c}}+\sqrt{\frac{h_c^3}{h_c+\lambda h_a}}\geq\frac{9r}{\sqrt{\lambda+1}},\text{ unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=h_a, y=h_b, z=h_c$ obținem:

$$\sqrt{\frac{h_a^3}{h_a+\lambda h_b}}+\sqrt{\frac{h_b^3}{h_b+\lambda h_c}}+\sqrt{\frac{h_c^3}{h_c+\lambda h_a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum h_a\geq\frac{1}{\sqrt{\lambda+1}}\cdot 9r=\frac{9r}{\sqrt{\lambda+1}}.$$

Aplicația 39

In ΔABC

$$\sqrt{\frac{r_a^3}{r_a+8r_b}}+\sqrt{\frac{r_b^3}{r_b+8r_c}}+\sqrt{\frac{r_c^3}{r_c+8r_a}}\geq 3r.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=r_a, y=r_b, z=r_c$ obținem:

$$\sqrt{\frac{r_a^3}{r_a+8r_b}}+\sqrt{\frac{r_b^3}{r_b+8r_c}}+\sqrt{\frac{r_c^3}{r_c+8r_a}}\overset{Lema}{\geq}\frac{1}{3}\sum r_a\geq\frac{1}{3}\cdot 9r=3r.$$

Aplicația 40

In ΔABC

$$\sqrt{\frac{r_a^3}{r_a+\lambda r_b}}+\sqrt{\frac{r_b^3}{r_b+\lambda r_c}}+\sqrt{\frac{r_c^3}{r_c+\lambda r_a}}\geq\frac{9r}{\sqrt{\lambda+1}},\text{ unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=h_a, y=h_b, z=h_c$ obținem:

$$\sqrt{\frac{r_a^3}{r_a+\lambda r_b}}+\sqrt{\frac{r_b^3}{r_b+\lambda r_c}}+\sqrt{\frac{r_c^3}{r_c+\lambda r_a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum r_a\geq\frac{1}{\sqrt{\lambda+1}}\cdot 9r=\frac{9r}{\sqrt{\lambda+1}}.$$

Aplicația 41

In ΔABC

$$\sqrt{\frac{m_a^3}{m_a+8m_b}}+\sqrt{\frac{m_b^3}{m_b+8m_c}}+\sqrt{\frac{m_c^3}{m_c+8m_a}}\geq 3r.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=m_a, y=m_b, z=m_c$ obținem:

$$\sqrt{\frac{r_a^3}{r_a+8r_b}}+\sqrt{\frac{r_b^3}{r_b+8r_c}}+\sqrt{\frac{r_c^3}{r_c+8r_a}}\overset{Lema}{\geq}\frac{1}{3}\sum r_a\geq\frac{1}{3}\cdot 9r=3r.$$

Aplicația 42

In ΔABC

$$\sqrt{\frac{m_a^3}{m_a+\lambda m_b}}+\sqrt{\frac{m_b^3}{m_b+\lambda m_c}}+\sqrt{\frac{m_c^3}{m_c+\lambda m_a}}\geq\frac{9r}{\sqrt{\lambda+1}},\text{ unde } \lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=m_a, y=m_b, z=m_c$ obținem:

$$\sqrt{\frac{m_a^3}{m_a+\lambda m_b}}+\sqrt{\frac{m_b^3}{m_b+\lambda m_c}}+\sqrt{\frac{m_c^3}{m_c+\lambda m_a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum m_a\geq\frac{1}{\sqrt{\lambda+1}}\cdot 9r=\frac{9r}{\sqrt{\lambda+1}}.$$

Aplicația 43

In ΔABC

$$\sqrt{\frac{w_a^3}{w_a+8w_b}}+\sqrt{\frac{w_b^3}{w_b+8w_c}}+\sqrt{\frac{w_c^3}{w_c+8w_a}}\geq 3r.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=w_a, y=w_b, z=w_c$ obținem:

$$\sqrt{\frac{w_a^3}{w_a+8w_b}}+\sqrt{\frac{w_b^3}{w_b+8w_c}}+\sqrt{\frac{w_c^3}{w_c+8w_a}}\overset{Lema}{\geq}\frac{1}{3}\sum w_a\geq\frac{1}{3}\cdot 9r=3r.$$

Aplicația 44

In ΔABC

$$\sqrt{\frac{w_a^3}{w_a+\lambda w_b}}+\sqrt{\frac{w_b^3}{w_b+\lambda w_c}}+\sqrt{\frac{w_c^3}{w_c+\lambda w_a}}\geq\frac{9r}{\sqrt{\lambda+1}}\text{, unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=w_a, y=w_b, z=w_c$ obținem:

$$\sqrt{\frac{w_a^3}{w_a+\lambda w_b}}+\sqrt{\frac{w_b^3}{w_b+\lambda w_c}}+\sqrt{\frac{w_c^3}{w_c+\lambda w_a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum w_a\geq\frac{1}{\sqrt{\lambda+1}}\cdot 9r=\frac{9r}{\sqrt{\lambda+1}}.$$

Aplicația 45

In ΔABC

$$\sqrt{\frac{s_a^3}{s_a+8s_b}}+\sqrt{\frac{s_b^3}{s_b+8s_c}}+\sqrt{\frac{s_c^3}{s_c+8s_a}}\geq 3r.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=s_a, y=s_b, z=s_c$ obținem:

$$\sqrt{\frac{s_a^3}{s_a+8s_b}}+\sqrt{\frac{s_b^3}{s_b+8s_c}}+\sqrt{\frac{s_c^3}{s_c+8s_a}}\overset{Lema}{\geq}\frac{1}{3}\sum s_a\geq\frac{1}{3}\cdot 9r=3r.$$

Aplicația 46

In ΔABC

$$\sqrt{\frac{s_a^3}{s_a+\lambda s_b}}+\sqrt{\frac{s_b^3}{s_b+\lambda s_c}}+\sqrt{\frac{s_c^3}{s_c+\lambda s_a}}\geq\frac{9r}{\sqrt{\lambda+1}}\text{, unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=s_a, y=s_b, z=s_c$ obținem:

$$\sqrt{\frac{s_a^3}{s_a+\lambda s_b}}+\sqrt{\frac{s_b^3}{s_b+\lambda s_c}}+\sqrt{\frac{s_c^3}{s_c+\lambda s_a}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum s_a\geq\frac{1}{\sqrt{\lambda+1}}\cdot 9r=\frac{9r}{\sqrt{\lambda+1}}.$$

Aplicația 47

In ΔABC

$$\sqrt{\frac{\tan^3\frac{A}{2}}{\tan\frac{A}{2}+8\tan\frac{B}{2}}}+\sqrt{\frac{\tan^3\frac{B}{2}}{\tan\frac{B}{2}+8\tan\frac{C}{2}}}+\sqrt{\frac{\tan^3\frac{C}{2}}{\tan\frac{C}{2}+8\tan\frac{A}{2}}}\geq\frac{\sqrt{3}}{3}.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\tan^3\frac{A}{2}}{\tan\frac{A}{2}+8\tan\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\tan\frac{A}{2}=\frac{1}{3}\cdot\frac{4R+r}{p}\overset{Doucet}{\geq}\frac{1}{3}\cdot\sqrt{3}=\frac{\sqrt{3}}{3}=Md.$$

Aplicația 48

In ΔABC

$$\sqrt{\frac{\tan^3\frac{A}{2}}{\tan\frac{A}{2}+\lambda\tan\frac{B}{2}}}+\sqrt{\frac{\tan^3\frac{B}{2}}{\tan\frac{B}{2}+\lambda\tan\frac{C}{2}}}+\sqrt{\frac{\tan^3\frac{C}{2}}{\tan\frac{C}{2}+\lambda\tan\frac{A}{2}}}\geq\frac{\sqrt{3}}{\sqrt{\lambda+1}},\text{ unde }\lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}$ obținem:

$$\sum\sqrt{\frac{\tan^3\frac{A}{2}}{\tan\frac{A}{2}+\lambda\tan\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum\tan\frac{A}{2}=\frac{1}{\sqrt{\lambda+1}}\cdot\frac{4R+r}{p}\overset{Doucet}{\geq}\frac{1}{\sqrt{\lambda+1}}\cdot\sqrt{3}=\frac{\sqrt{3}}{\sqrt{\lambda+1}}.$$

Aplicația 49

In ΔABC

$$\sqrt{\frac{\cot^3\frac{A}{2}}{\cot\frac{A}{2}+8\cot\frac{B}{2}}}+\sqrt{\frac{\cot^3\frac{B}{2}}{\cot\frac{B}{2}+8\cot\frac{C}{2}}}+\sqrt{\frac{\cot^3\frac{C}{2}}{\cot\frac{C}{2}+8\cot\frac{A}{2}}}\geq\sqrt{3}.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\cot\frac{A}{2}, y=\cot\frac{B}{2}, z=\cot\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\cot^3\frac{A}{2}}{\cot\frac{A}{2}+8\cot\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\cot\frac{A}{2}=\frac{1}{3}\cdot\frac{p}{r}\overset{Mitrinovic}{\geq}\frac{1}{3}\cdot3\sqrt{3}=\sqrt{3}=Md.$$

Aplicația 50

In ΔABC

$$\sqrt{\frac{\cot^3\frac{A}{2}}{\cot\frac{A}{2}+\lambda\cot\frac{B}{2}}}+\sqrt{\frac{\cot^3\frac{B}{2}}{\cot\frac{B}{2}+\lambda\cot\frac{C}{2}}}+\sqrt{\frac{\cot^3\frac{C}{2}}{\cot\frac{C}{2}+\lambda\cot\frac{A}{2}}}\geq\frac{3\sqrt{3}}{\sqrt{\lambda+1}},\ \text{unde}\ \lambda\geq0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\cot\frac{A}{2}, y=\cot\frac{B}{2}, z=\cot\frac{C}{2}$ obținem:

$$\sum\sqrt{\frac{\cot^3\frac{A}{2}}{\cot\frac{A}{2}+\lambda\cot\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum\cot\frac{A}{2}=\frac{1}{\sqrt{\lambda+1}}\cdot\frac{p}{r}\overset{Mitrinovic}{\geq}\frac{1}{\sqrt{\lambda+1}}\cdot 3\sqrt{3}=\frac{3\sqrt{3}}{\sqrt{\lambda+1}}.$$

Aplicația 51

In ΔABC

$$\sqrt{\frac{\sin^3\frac{A}{2}}{\sin\frac{A}{2}+8\sin\frac{B}{2}}}+\sqrt{\frac{\sin^3\frac{B}{2}}{\sin\frac{B}{2}+8\sin\frac{C}{2}}}+\sqrt{\frac{\sin^3\frac{C}{2}}{\sin\frac{C}{2}+8\sin\frac{A}{2}}}\geq\frac{1}{3}\left(1+\frac{r}{R}\right).$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, z=\sin\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\sin^3\frac{A}{2}}{\sin\frac{A}{2}+8\sin\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\sin\frac{A}{2}\geq\frac{1}{3}\cdot\left(1+\frac{r}{R}\right)=Md,$$

care rezultă din: **Propoziția1**: $\sum\sin\frac{A}{2}\geq 1+\frac{r}{R}$, adevărată din:

$$1+\frac{r}{R}=\sum\cos A=\sum\cos\frac{B+C}{2}\cos\frac{B-C}{2}\leq\sum\cos\frac{B+C}{2}=\sum\sin\frac{A}{2}.$$

Aplicația 52

In ΔABC

$$\sqrt{\frac{\sin^3\frac{A}{2}}{\sin\frac{A}{2}+\lambda\sin\frac{B}{2}}}+\sqrt{\frac{\sin^3\frac{B}{2}}{\sin\frac{B}{2}+\lambda\sin\frac{C}{2}}}+\sqrt{\frac{\sin^3\frac{C}{2}}{\sin\frac{C}{2}+\lambda\sin\frac{A}{2}}}\geq\frac{1}{\sqrt{\lambda+1}}\left(1+\frac{r}{R}\right),\ \text{unde}\ \lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, z=\sin\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\sin^3\frac{A}{2}}{\sin\frac{A}{2}+\lambda\sin\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum\sin\frac{A}{2}\geq\frac{1}{\sqrt{\lambda+1}}\cdot\left(1+\frac{r}{R}\right)=Md,$$

care rezultă din: **Propoziția1**, vezi Aplicatia51.

Aplicația 53

In ΔABC

$$\sqrt{\frac{\cos^3\frac{A}{2}}{\cos\frac{A}{2}+8\cos\frac{B}{2}}}+\sqrt{\frac{\cos^3\frac{B}{2}}{\cos\frac{B}{2}+8\cos\frac{C}{2}}}+\sqrt{\frac{\cos^3\frac{C}{2}}{\cos\frac{C}{2}+8\cos\frac{A}{2}}}\geq\frac{1}{3}\left(1+\frac{r}{R}\right).$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\cos^3\frac{A}{2}}{\cos\frac{A}{2}+8\cos\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\cos\frac{A}{2}\geq\frac{1}{3}\cdot\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)=\frac{\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)=Md,$$

care rezultă din: **Propoziția2**: $\sum\cos\frac{A}{2}\geq\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)$, adevărată din:

Se aplică inegalitatea lui Popoviciu funcției concave $f:(0,\pi)\to R, f(x)=\sin x$.

Avem: $\frac{\sin A+\sin B+\sin C}{3}+\sin\frac{A+B+C}{3}\leq\frac{2}{3}\sum\sin\frac{B+C}{2}\Leftrightarrow\frac{1}{3}\cdot\frac{p}{R}+\frac{\sqrt{3}}{2}\leq\frac{2}{3}\sum\cos\frac{A}{2}\Leftrightarrow$

$\Leftrightarrow\sum\cos\frac{A}{2}\geq\frac{p}{2R}+\frac{3\sqrt{3}}{4}$.

Apoi: $\sum\cos\frac{A}{2}\geq\frac{p}{2R}+\frac{3\sqrt{3}}{4}\overset{Mitrinovic}{\geq}\frac{3r\sqrt{3}}{2R}+\frac{3\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)$.

Aplicația 54

In ΔABC

$$\sqrt{\frac{\cos^3\frac{A}{2}}{\cos\frac{A}{2}+\lambda\cos\frac{B}{2}}}+\sqrt{\frac{\cos^3\frac{B}{2}}{\cos\frac{B}{2}+\lambda\cos\frac{C}{2}}}+\sqrt{\frac{\cos^3\frac{C}{2}}{\cos\frac{C}{2}+\lambda\cos\frac{A}{2}}}\ge\frac{3\sqrt{3}}{\sqrt{\lambda+1}}\left(\frac{1}{4}+\frac{r}{2R}\right),\ \lambda\ge 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\ge\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\cos^3\frac{A}{2}}{\cos\frac{A}{2}+\lambda\cos\frac{B}{2}}}\overset{Lema}{\ge}\frac{1}{\sqrt{\lambda+1}}\sum\cos\frac{A}{2}\ge\frac{1}{\sqrt{\lambda+1}}\cdot\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)=Md,$$

care rezultă din **Propoziția2:** $\sum\cos\frac{A}{2}\ge\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)$,vezi Aplicația53.

Aplicația 55

In ΔABC

$$\sqrt{\frac{\sec^3\frac{A}{2}}{\sec\frac{A}{2}+8\sec\frac{B}{2}}}+\sqrt{\frac{\sec^3\frac{B}{2}}{\sec\frac{B}{2}+8\sec\frac{C}{2}}}+\sqrt{\frac{\sec^3\frac{C}{2}}{\sec\frac{C}{2}+8\sec\frac{A}{2}}}\ge\frac{\sqrt{3}}{3}\left(1+\frac{2r}{R}\right).$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\ge\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\ge}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\sec\frac{A}{2}, y=\sec\frac{B}{2}, z=\sec\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\sec^3\frac{A}{2}}{\sec\frac{A}{2}+8\sec\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\sec\frac{A}{2}\geq\frac{1}{3}\cdot 2\sqrt{3}\left(\frac{1}{2}+\frac{r}{R}\right)=\frac{\sqrt{3}}{3}\left(1+\frac{2r}{R}\right)=Md$$, care rezultă

din: $\sum\sec\frac{A}{2}\geq 2\sqrt{3}$, adevărată din: $\sum\sec\frac{A}{2}=\sum\frac{1}{\cos\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\cos\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}$.

Aplicația 56

In ΔABC

$$\sqrt{\frac{\sec^3\frac{A}{2}}{\sec\frac{A}{2}+\lambda\sec\frac{B}{2}}}+\sqrt{\frac{\sec^3\frac{B}{2}}{\sec\frac{B}{2}+\lambda\sec\frac{C}{2}}}+\sqrt{\frac{\sec^3\frac{C}{2}}{\sec\frac{C}{2}+\lambda\sec\frac{A}{2}}}\geq\frac{2\sqrt{3}}{\sqrt{\lambda+1}},\ \lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\sec\frac{A}{2}, y=\sec\frac{B}{2}, z=\sec\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\sec^3\frac{A}{2}}{\sec\frac{A}{2}+\lambda\sec\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum\sec\frac{A}{2}\geq\frac{1}{\sqrt{\lambda+1}}\cdot 2\sqrt{3}=\frac{2\sqrt{3}}{\sqrt{\lambda+1}}=Md$$, care rezultă

din: $\sum\sec\frac{A}{2}\geq 2\sqrt{3}$, adevărată din: $\sum\sec\frac{A}{2}=\sum\frac{1}{\cos\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\cos\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}$.

Aplicația 57

In ΔABC

$$\sqrt{\frac{\csc^3\frac{A}{2}}{\csc\frac{A}{2}+8\csc\frac{B}{2}}}+\sqrt{\frac{\csc^3\frac{B}{2}}{\csc\frac{B}{2}+8\csc\frac{C}{2}}}+\sqrt{\frac{\csc^3\frac{C}{2}}{\csc\frac{C}{2}+8\csc\frac{A}{2}}}\geq 2.$$

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+8y}}+\sqrt{\frac{y^3}{y+8z}}+\sqrt{\frac{z^3}{z+8x}}\geq\frac{1}{3}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+8y}}=\sum\frac{x^{\frac{3}{2}}}{(x+8y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+8y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left(9\sum x\right)^{\frac{1}{2}}}=\frac{1}{3}\sum x.$$

Folosind **Lema** pentru $x=\csc\frac{A}{2}, y=\csc\frac{B}{2}, z=\csc\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\csc^3\frac{A}{2}}{\csc\frac{A}{2}+8\csc\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{3}\sum\csc\frac{A}{2}\geq\frac{1}{3}\cdot 6=2=Md$$, care rezultă din: $\sum\csc\frac{A}{2}\geq 6$,

adevărată din: $\sum\csc\frac{A}{2}=\sum\frac{1}{\sin\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\sin\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3}{2}}=6$.

Aplicația 58

In ΔABC

$$\sqrt{\frac{\csc^3\frac{A}{2}}{\csc\frac{A}{2}+\lambda\csc\frac{B}{2}}}+\sqrt{\frac{\csc^3\frac{B}{2}}{\csc\frac{B}{2}+\lambda\csc\frac{C}{2}}}+\sqrt{\frac{\csc^3\frac{C}{2}}{\csc\frac{C}{2}+\lambda\csc\frac{A}{2}}}\geq\frac{6}{\sqrt{\lambda+1}},\ \lambda\geq 0.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{x+\lambda y}}+\sqrt{\frac{y^3}{y+\lambda z}}+\sqrt{\frac{z^3}{z+\lambda x}}\geq\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Demonstrație.

$$\sum\sqrt{\frac{x^3}{x+\lambda y}}=\sum\frac{x^{\frac{3}{2}}}{(x+\lambda y)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum(x+\lambda y)\right]^{\frac{1}{2}}}=\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left((\lambda+1)\sum x\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{\lambda+1}}\sum x.$$

Folosind **Lema** pentru $x=\csc\frac{A}{2}, y=\csc\frac{B}{2}, z=\csc\frac{C}{2}$ obținem:

$$Ms=\sum\sqrt{\frac{\csc^3\frac{A}{2}}{\csc\frac{A}{2}+\lambda\csc\frac{B}{2}}}\overset{Lema}{\geq}\frac{1}{\sqrt{\lambda+1}}\sum\csc\frac{A}{2}\geq\frac{1}{\sqrt{\lambda+1}}\cdot 6=\frac{6}{\sqrt{\lambda+1}}=Md$$, care rezultă din:

$\sum\csc\frac{A}{2}\geq 6$, adevărată din: $\sum\csc\frac{A}{2}=\sum\frac{1}{\sin\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\sin\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3}{2}}=6$.

Aplicația 59

If $x,y,z>0$ then

$$\sqrt{\frac{x^3}{y^3+8xyz}}+\sqrt{\frac{y^3}{z^3+8xyz}}+\sqrt{\frac{z^3}{x^3+8xyz}}\geq 1.$$

Mathematical Inequalities, Imad Zak, Lebanon, 5/6/21

Soluție.

$$Ms=\sum\sqrt{\frac{x^3}{y^3+8xyz}}=\sum\frac{x^{\frac{3}{2}}}{\left(y^3+8xyz\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum\left(y^3+8xyz\right)\right]^{\frac{1}{2}}}\overset{(1)}{\geq}1=Md\text{ , unde (1) }\Leftrightarrow$$

$$\Leftrightarrow\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum\left(y^3+8xyz\right)\right]^{\frac{1}{2}}}\geq1\Leftrightarrow\left(\sum x\right)^{\frac{3}{2}}\geq\left[\sum\left(y^3+8xyz\right)\right]^{\frac{1}{2}}\Leftrightarrow\left(\sum x\right)^3\geq\sum\left(y^3+8xyz\right)\Leftrightarrow$$

$$\Leftrightarrow\sum x^3+3\prod(y+z)\geq\sum x^3+24xyz\Leftrightarrow\prod(y+z)\geq8xyz\text{ ,(Cesaro).}$$

Egalitatea are loc dacă și numai dacă $x=y=z$.

Aplicația 60

If $x,y,z>0$ and $\lambda\geq8$ then

$$\sqrt{\frac{x^3}{y^3+\lambda xyz}}+\sqrt{\frac{y^3}{z^3+\lambda xyz}}+\sqrt{\frac{z^3}{x^3+\lambda xyz}}\geq\frac{3}{\sqrt{\lambda+1}}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\sqrt{\frac{x^3}{y^3+\lambda xyz}}=\sum\frac{x^{\frac{3}{2}}}{\left(y^3+\lambda xyz\right)^{\frac{1}{2}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum\left(y^3+\lambda xyz\right)\right]^{\frac{1}{2}}}\overset{(1)}{\geq}\frac{3}{\sqrt{\lambda+1}}=Md\text{ , unde (1) }\Leftrightarrow$$

$$\Leftrightarrow\frac{\left(\sum x\right)^{\frac{3}{2}}}{\left[\sum\left(y^3+\lambda xyz\right)\right]^{\frac{1}{2}}}\geq\frac{3}{\sqrt{\lambda+1}}\Leftrightarrow\sqrt{\lambda+1}\left(\sum x\right)^{\frac{3}{2}}\geq3\left[\sum\left(y^3+\lambda xyz\right)\right]^{\frac{1}{2}}\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\left(\sum x\right)^3\geq9\sum\left(y^3+\lambda xyz\right)\Leftrightarrow(\lambda+1)\sum x^3+3(\lambda+1)\prod(y+z)\geq9\sum x^3+27\lambda xyz\Leftrightarrow$$

$\Leftrightarrow(\lambda-8)\sum x^3+3\left[(\lambda+1)\prod(y+z)-9\lambda xyz\right]\geq0$, care rezultă din condiția din ipoteză $\lambda\geq8$ și inegalitatea lui Cesaro $\prod(y+z)\geq8xyz$. Este suficient să arătăm că:

$$(\lambda-8)\sum x^3+3\left[(\lambda+1)\cdot8xyz-9\lambda xyz\right]\geq0\Leftrightarrow(\lambda-8)\sum x^3+3xyz\left[8(\lambda+1)-9\lambda\right]\geq0\Leftrightarrow$$

$\Leftrightarrow(\lambda-8)\left(\sum x^3-3xyz\right)\geq0$, adevărată din $\lambda\geq8$ și $\sum x^3\geq3xyz$,(AM-GM).

Egalitatea are loc dacă și numai dacă $x=y=z$.

Aplicația 61

If $x,y,z>0$ such that $xy+yz+zx=\sqrt{3}$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\sqrt{3}.$$

Mathematical Inequalities, Imad Zak, Lebanon, 7/6/21

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** este suficient să arătăm că: $\left(\sum x\right)^{\frac{2}{3}}\geq\sqrt{3}$, care rezultă din:

$$\left(\sum x\right)^{\frac{2}{3}}=\left(\sum x^2\right)^{\frac{1}{3}}\geq\left(3\sum yz\right)^{\frac{1}{3}}=\left(3\sqrt{3}\right)^{\frac{1}{3}}=\sqrt{3}=Md\ .$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{\sqrt[4]{3}}$.

Aplicația 62

In ΔABC

$$\sqrt[3]{\frac{a^4}{b^2+c(a+b)}}+\sqrt[3]{\frac{b^4}{c^2+a(b+c)}}+\sqrt[3]{\frac{c^4}{a^2+b(c+a)}}\geq 3\sqrt[3]{4r^2}\ .$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=a,y=b,z=c$ este suficient să arătăm că: $\left(\sum a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{4r^2}$, din:

$\left(\sum a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{4r^2}\Leftrightarrow\left(2p\right)^{\frac{2}{3}}\geq 3\sqrt[3]{4r^2}$, adevărată din inegalitatea lui Mitrinovic: $p\geq 3r\sqrt{3}$.

Rămâne să arătăm că:

$\left(2\cdot 3r\sqrt{3}\right)^{\frac{2}{3}}\geq 3\sqrt[3]{4r^2}\Leftrightarrow\left(2\cdot 3r\sqrt{3}\right)^{\frac{2}{3}}\geq 3\sqrt[3]{4r^2}\Leftrightarrow\left(6r\sqrt{3}\right)^2\geq 27\cdot 4r^2\Leftrightarrow 108r^2\geq 108r^2$, evident.

Aplicația 63

In ΔABC

$$\sqrt[3]{\frac{h_a^4}{h_b^2+h_c\left(h_a+h_b\right)}}+\sqrt[3]{\frac{h_b^4}{h_c^2+h_a\left(h_b+h_c\right)}}+\sqrt[3]{\frac{h_c^4}{h_a^2+h_b\left(h_c+h_a\right)}}\geq 3\sqrt[3]{3r^2}\ .$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x = h_a, y = h_b, z = h_c$ este suficient să arătăm că: $\left(\sum h_a\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2}$, care rezultă din: $\sum h_a \geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2} \Leftrightarrow (9r)^2 \geq 27\cdot 3r^2 \Leftrightarrow 81r^2 \geq 81r^2$, evident.

Aplicația 64

In ΔABC

$$\sqrt[3]{\frac{r_a^4}{r_b^2+r_c(r_a+r_b)}}+\sqrt[3]{\frac{r_b^4}{r_c^2+r_a(r_b+r_c)}}+\sqrt[3]{\frac{r_c^4}{r_a^2+r_b(r_c+r_a)}} \geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x = r_a, y = r_b, z = r_c$ este suficient să arătăm că: $\left(\sum r_a\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2}$, care rezultă din: $\sum r_a \geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}} \geq 3\sqrt[3]{3r^2} \Leftrightarrow (9r)^2 \geq 27\cdot 3r^2 \Leftrightarrow 81r^2 \geq 81r^2$, evident.

Aplicația 65

In ΔABC

$$\sqrt[3]{\frac{m_a^4}{m_b^2+m_c\left(m_a+m_b\right)}}+\sqrt[3]{\frac{m_b^4}{m_c^2+m_a\left(m_b+m_c\right)}}+\sqrt[3]{\frac{m_c^4}{m_a^2+m_b\left(m_c+m_a\right)}}\geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

Folosind **Lema** pentru $x=m_a, y=m_b, z=m_c$ este suficient să arătăm că: $\left(\sum m_a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}$,

care rezultă din: $\sum m_a\geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}\Leftrightarrow(9r)^2\geq 27\cdot 3r^2\Leftrightarrow 81r^2\geq 81r^2$, evident.

Aplicația 66

In ΔABC

$$\sqrt[3]{\frac{w_a^4}{w_b^2+w_c\left(w_a+w_b\right)}}+\sqrt[3]{\frac{w_b^4}{w_c^2+w_a\left(w_b+w_c\right)}}+\sqrt[3]{\frac{m_c^4}{w_a^2+w_b\left(w_c+w_a\right)}}\geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=w_a, y=w_b, z=w_c$ este suficient să arătăm că: $\left(\sum w_a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}$,

care rezultă din: $\sum w_a\geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}\Leftrightarrow(9r)^2\geq 27\cdot 3r^2\Leftrightarrow 81r^2\geq 81r^2$, evident.

Aplicația 67

In ΔABC

$$\sqrt[3]{\frac{s_a^4}{s_b^2+s_c\left(s_a+s_b\right)}}+\sqrt[3]{\frac{s_b^4}{s_c^2+s_a\left(s_b+s_c\right)}}+\sqrt[3]{\frac{s_c^4}{s_a^2+s_b\left(s_c+s_a\right)}}\geq 3\sqrt[3]{3r^2}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

Folosind **Lema** pentru $x=s_a, y=s_b, z=s_c$ este suficient să arătăm că: $\left(\sum s_a\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}$,
care rezultă din: $\sum s_a\geq\sum h_a\geq 9r$. Rămâne să arătăm că:

$\left(\sum 9r\right)^{\frac{2}{3}}\geq 3\sqrt[3]{3r^2}\Leftrightarrow(9r)^2\geq 27\cdot 3r^2\Leftrightarrow 81r^2\geq 81r^2$, evident.

Aplicația 68

In ΔABC

$$\sum\sqrt[3]{\frac{\tan^4\frac{A}{2}}{\tan^2\frac{B}{2}+\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)}}\geq\sqrt[3]{3}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\tan\frac{A}{2}\right)^{\frac{2}{3}}\geq\sqrt[3]{3}$, care rezultă din: $\left(\sum\tan\frac{A}{2}\right)^{\frac{2}{3}}=\left(\frac{4R+r}{p}\right)^{\frac{2}{3}}\overset{Doucet}{\geq}\left(\sqrt{3}\right)^{\frac{2}{3}}=\sqrt[3]{3}$.

Aplicația 69

In ΔABC

$$\sum \sqrt[3]{\frac{\cot^4 \frac{A}{2}}{\cot^2 \frac{B}{2} + \cot \frac{C}{2}\left(\cot \frac{A}{2} + \cot \frac{B}{2}\right)}} \geq \sqrt[3]{3}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} + \sqrt[3]{\frac{y^4}{z^2 + x(y+z)}} + \sqrt[3]{\frac{z^4}{x^2 + y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum \sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} = \sum \frac{x^{\frac{4}{3}}}{\left[y^2 + z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum \left(x^2 + z(x+y)\right)\right]^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2 + 2\sum yz\right)^{\frac{1}{3}}} =$$

$$= \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2 + 2\sum yz\right)^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}} = \left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x = \cot \frac{A}{2}, y = \cot \frac{B}{2}, z = \cot \frac{C}{2}$ este suficient să arătăm că:

$\left(\sum \cot \frac{A}{2}\right)^{\frac{2}{3}} \geq 3$, care rezultă din: $\left(\sum \cot \frac{A}{2}\right)^{\frac{2}{3}} = \left(\frac{p}{3}\right)^{\frac{2}{3}} \overset{Mitrinovic}{\geq} \left(3\sqrt{3}\right)^{\frac{2}{3}} = \sqrt[3]{27} = 3.$

Aplicația 70

In ΔABC

$$\sum \sqrt[3]{\frac{\sin^4 \frac{A}{2}}{\sin^2 \frac{B}{2} + \sin \frac{C}{2}\left(\sin \frac{A}{2} + \sin \frac{B}{2}\right)}} \geq \left(1 + \frac{r}{R}\right)^{\frac{2}{3}}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x, y, z > 0$ then

$$\sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} + \sqrt[3]{\frac{y^4}{z^2 + x(y+z)}} + \sqrt[3]{\frac{z^4}{x^2 + y(z+x)}} \geq \left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum \sqrt[3]{\frac{x^4}{y^2 + z(x+y)}} = \sum \frac{x^{\frac{4}{3}}}{\left[y^2 + z(x+y)\right]^{\frac{1}{3}}} \overset{Radon}{\geq} \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum \left(x^2 + z(x+y)\right)\right]^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2 + 2\sum yz\right)^{\frac{1}{3}}} =$$

$$= \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2 + 2\sum yz\right)^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}} = \frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}} = \left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, z=\sin\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\sin\frac{A}{2}\right)^{\frac{2}{3}}\geq\left(1+\frac{r}{R}\right)^{\frac{2}{3}}$, care rezultă din: **Propoziția1**, vezi Aplicatia51.

Aplicația 71

In ΔABC

$$\sum\sqrt[3]{\frac{\cos^4\frac{A}{2}}{\cos^2\frac{B}{2}+\cos\frac{C}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)}}\geq 3\left(\frac{1}{4}+\frac{r}{2R}\right)^{\frac{2}{3}}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\cos\frac{A}{2}\right)^{\frac{2}{3}}\geq 3\left(\frac{1}{4}+\frac{r}{2R}\right)^{\frac{2}{3}}$, care rezultă din: $\sum\cos\frac{A}{2}\geq 3\sqrt{3}\left(\frac{1}{2}+\frac{r}{R}\right)$, adevărată din:

Propoziția2: $\sum\cos\frac{A}{2}\geq\frac{3\sqrt{3}}{2}\left(\frac{1}{2}+\frac{r}{R}\right)$,vezi Aplicația53.

Aplicația 72

In ΔABC

$$\sum\sqrt[3]{\frac{\sec^4\frac{A}{2}}{\sec^2\frac{B}{2}+\sec\frac{C}{2}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)}}\geq\sqrt[3]{12}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\sec\frac{A}{2}, y=\sec\frac{B}{2}, z=\sec\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\sec\frac{A}{2}\right)^{\frac{2}{3}}\geq\left(2\sqrt{3}\right)^{\frac{2}{3}}=\sqrt[3]{12}$, care rezultă din: $\sum\sec\frac{A}{2}\geq 2\sqrt{3}$, adevărată din:

$$\sum\sec\frac{A}{2}=\sum\frac{1}{\cos\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\cos\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}.$$

Aplicația 73

In ΔABC

$$\sum\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}}\geq\sqrt[3]{36}.$$

Marin Chirciu

Soluție.

Demonstrăm **Lema:**

If $x,y,z>0$ then

$$\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}+\sqrt[3]{\frac{y^4}{z^2+x(y+z)}}+\sqrt[3]{\frac{z^4}{x^2+y(z+x)}}\geq\left(\sum x\right)^{\frac{2}{3}}.$$

Demonstrație.

$$\sum\sqrt[3]{\frac{x^4}{y^2+z(x+y)}}=\sum\frac{x^{\frac{4}{3}}}{\left[y^2+z(x+y)\right]^{\frac{1}{3}}}\overset{Radon}{\geq}\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\sum\left(x^2+z(x+y)\right)\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=$$

$$=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x^2+2\sum yz\right)^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left[\left(\sum x\right)^2\right]^{\frac{1}{3}}}=\frac{\left(\sum x\right)^{\frac{4}{3}}}{\left(\sum x\right)^{\frac{2}{3}}}=\left(\sum x\right)^{\frac{2}{3}}.$$

Folosind **Lema** pentru $x=\csc\frac{A}{2}, y=\csc\frac{B}{2}, z=\csc\frac{C}{2}$ este suficient să arătăm că:

$\left(\sum\csc\frac{A}{2}\right)^{\frac{2}{3}}\geq(6)^{\frac{2}{3}}=\sqrt[3]{36}$, care rezultă din: $\sum\csc\frac{A}{2}\geq 6$, adevărată din:

$$\sum\csc\frac{A}{2}=\sum\frac{1}{\sin\frac{A}{2}}\overset{AHM}{\geq}\frac{9}{\sum\sin\frac{A}{2}}\overset{Jensen}{\geq}\frac{9}{\frac{3}{2}}=6.$$

Aplicația 74

If $a,b,c,x,y,z>0$ such that $xyz=1$ and $a^2+b^2+c^2=3$ then

$$\sum\frac{1}{(a+b)^2\left(x^4+x^2+1\right)^3}\geq\frac{1}{36}.$$

Nguyen Van Canh

Soluție.

$$Ms=\sum\frac{1}{(a+b)^2\left(x^4+x^2+1\right)^3}=\sum\frac{\frac{1}{\left(x^4+x^2+1\right)^3}}{(a+b)^2}\overset{Radon}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\sum(a+b)^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\left(\sum(a+b)\right)^2}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\left(2\sum a\right)^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{4\left(\sum a\right)^2}\overset{CS}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{4\cdot3\sum a^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{4\cdot3\cdot3}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{4\cdot3\cdot3}=\frac{1}{36}\sum\frac{1}{\left(x^4+x^2+1\right)^3}\overset{(1)}{\geq}\frac{1}{36}=Md,$$

unde (1) $\Leftrightarrow\sum\frac{1}{\left(x^4+x^2+1\right)^3}\geq1\Leftrightarrow\sum\frac{1}{x^4+x^2+1}\geq1$, care rezultă din substituția:

$x=\frac{\sqrt{BC}}{A},y=\frac{\sqrt{CA}}{B},z=\frac{\sqrt{AB}}{C}$, $A,B,C>0$, $xyz=1$.

Obținem:

$$\sum\frac{1}{\left(\frac{\sqrt{BC}}{A}\right)^4+\left(\frac{\sqrt{BC}}{A}\right)^2+1}=\sum\frac{1}{\frac{B^2C^2}{A^4}+\frac{BC}{A^2}+1}=\sum\frac{A^4}{A^4+A^2BC+B^2C^2}\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\frac{\left(\sum A^2\right)^2}{\sum\left(A^4+A^2BC+B^2C^2\right)}=\frac{\sum A^4+2\sum B^2C^2}{\sum A^4+\sum A^2BC+\sum B^2C^2}\overset{(2)}{\geq}1,$$

unde (1) $\Leftrightarrow\frac{\sum A^4+2\sum B^2C^2}{\sum A^4+\sum A^2BC+\sum B^2C^2}\geq1\Leftrightarrow\sum B^2C^2\geq\sum A^2BC$,

care rezultă din $\sum X^2\geq\sum YZ$, cu $X=BC,Y=CA,Z=AB$.

Egalitatea are loc dacă și numai dacă $a=b=c=x=y=z=1$.

Aplicația 75

If $a,b,c,x,y,z>0$ such that $xyz=1$, $a^2+b^2+c^2=3$ and $\lambda\geq0$ then

$$\sum\frac{1}{(b+\lambda c)^2\left(x^4+x^2+1\right)^3}\geq\frac{1}{9(\lambda+1)^2}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{(b+\lambda c)^2\left(x^4+x^2+1\right)^3}=\sum\frac{\frac{1}{\left(x^4+x^2+1\right)^3}}{(b+\lambda c)^2}\overset{Radon}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\sum(b+\lambda c)^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\left(\sum(b+\lambda c)\right)^2}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{\left((\lambda+1)\sum a\right)^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{(\lambda+1)^2\left(\sum a\right)^2}\overset{CS}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{(\lambda+1)^2\cdot 3\sum a^2}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{(\lambda+1)^2\cdot 3\cdot 3}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^3}}{(\lambda+1)^2\cdot 3\cdot 3}=\frac{1}{9(\lambda+1)^2}\sum\frac{1}{\left(x^4+x^2+1\right)^3}\overset{(1)}{\geq}\frac{1}{9(\lambda+1)^2}=Md\,,$$

unde (1) $\Leftrightarrow \sum\frac{1}{\left(x^4+x^2+1\right)^3}\geq 1 \Leftrightarrow \sum\frac{1}{x^4+x^2+1}\geq 1$, care rezultă din substituția:

$x^2=X, y^2=Y, z^2=Z$, $Z,Y,Z>0$, $XYZ=1$.

Inegalitatea $\sum\frac{1}{x^4+x^2+1}\geq 1$ se scrie $\sum\frac{1}{X^2+X+1}\geq 1 \Leftrightarrow \sum X^2\geq\sum YZ$, evident.

Egalitatea are loc dacă și numai dacă $a=b=c=x=y=z=1$.

Aplicația 76

If $a,b,c,x,y,z>0$ such that $xyz=1$, $a^n+b^n+c^n=3$ and $\lambda\geq 0$, $n\in\mathbf{N}, n\geq 2$ then

$$\sum\frac{1}{(b+\lambda c)^n\left(x^4+x^2+1\right)^{n+1}}\geq\frac{1}{3^n(\lambda+1)^n}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{(b+\lambda c)^n\left(x^4+x^2+1\right)^{n+1}}=\sum\frac{\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{(b+\lambda c)^n}\overset{Radon}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{\sum(b+\lambda c)^n}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{\left(\sum(b+\lambda c)\right)^n}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{\left((\lambda+1)\sum a\right)^n}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{(\lambda+1)^n\left(\sum a\right)^n}\overset{CS}{\geq}\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^{n-1}\sum a^n}=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^{n-1}\cdot 3}=$$

$$=\frac{\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^n}=\frac{1}{3^n(\lambda+1)^n}\sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}\overset{(1)}{\geq}\frac{1}{3^n(\lambda+1)^n}=Md\,,$$

unde (1) $\Leftrightarrow \sum\frac{1}{\left(x^4+x^2+1\right)^{n+1}}\geq 1 \Leftrightarrow \sum\frac{1}{x^4+x^2+1}\geq 1$, care rezultă din substituția:

$x^2=X, y^2=Y, z^2=Z$, $Z,Y,Z>0$, $XYZ=1$.

Inegalitatea $\sum\frac{1}{x^4+x^2+1}\geq 1$ se scrie $\sum\frac{1}{X^2+X+1}\geq 1 \Leftrightarrow \sum X^2\geq\sum YZ$, evident.

Egalitatea are loc dacă și numai dacă $a=b=c=x=y=z=1$.

Aplicația 77

If $a,b,c,x,y,z>0$ such that $xyz=1$, $a^n+b^n+c^n=3$ and $\lambda\geq 0$, $n\in\mathbf{N}, n\geq 2$ then

$$\sum\frac{1}{(b+\lambda c)^n\left(x^2+x+1\right)^{n+1}}\geq\frac{1}{3^n(\lambda+1)^n}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{(b+\lambda c)^n\left(x^2+x+1\right)^{n+1}}=\sum\frac{\frac{1}{\left(x^2+x+1\right)^{n+1}}}{(b+\lambda c)^n}\overset{Radon}{\geq}\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{\sum(b+\lambda c)^n}=\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{\left(\sum(b+\lambda c)\right)^n}=$$

$$=\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{\left((\lambda+1)\sum a\right)^n}=\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{(\lambda+1)^n\left(\sum a\right)^n}\overset{CS}{\geq}\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^{n-1}\sum a^n}=\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^{n-1}\cdot 3}=$$

$$=\frac{\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}}{(\lambda+1)^n\cdot 3^n}=\frac{1}{3^n(\lambda+1)^n}\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}\overset{(1)}{\geq}\frac{1}{3^n(\lambda+1)^n}=Md,$$

unde (1) $\Leftrightarrow\sum\frac{1}{\left(x^2+x+1\right)^{n+1}}\geq 1\Leftrightarrow\sum\frac{1}{x^2+x+1}\geq 1\Leftrightarrow\sum x^2\geq\sum yz$, evident.

Egalitatea are loc dacă și numai dacă $a=b=c=x=y=z=1$.

Aplicația 78

In ΔABC

$$\frac{a}{(b+c)^3h_a^2}+\frac{b}{(c+a)^3h_b^2}+\frac{c}{(a+b)^3h_c^2}\geq\frac{3}{32F^2}.$$

D.M.Bătinețu-Giurgiu & Dan Nănuți

Soluție.

Aplicația 79

In ΔABC

$$\frac{a}{(b+c)^{n+1}h_a^n}+\frac{b}{(c+a)^{n+1}h_b^n}+\frac{c}{(a+b)^{n+1}h_c^n}\geq\frac{3}{2(4F)^n},\ n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+c)^{n+1}h_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)^{n+1}}}{a^nh_a^n}=\sum\frac{\left(\frac{a}{b+c}\right)^{n+1}}{(2F)^n}\overset{Holder}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{3^n(2F)^n}\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{3^n(2F)^n}=\frac{3}{2(4F)^n}=Md$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația 80

In ΔABC

$$\frac{a}{(b+\lambda c)^{n+1}h_a^n}+\frac{b}{(c+\lambda a)^{n+1}h_b^n}+\frac{c}{(a+\lambda b)^{n+1}h_c^n}\geq\frac{3}{(\lambda+1)^{n+1}(2F)^n},\lambda\geq 0,\ n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+\lambda c)^{n+1}h_a^n}=\sum\frac{\frac{a^{n+1}}{(b+\lambda c)^{n+1}}}{a^nh_a^n}=\sum\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(2F)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum 2F\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{3^n(2F)^n}=\frac{3}{(\lambda+1)^{n+1}(2F)^n}=Md\ .$$

Aplicația 81

In ΔABC

$$\frac{a}{(b+c)^{n+1}r_a^n}+\frac{b}{(c+a)^{n+1}r_b^n}+\frac{c}{(a+b)^{n+1}r_c^n}\geq\frac{3}{2}\left(\frac{r}{pR^2}\right)^n,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+c)^{n+1}r_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)^{n+1}}}{a^nr_a^n}=\sum\frac{\left(\frac{a}{b+c}\right)^{n+1}}{(ar_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{\left(\sum ar_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{\left(2p(2R-r)\right)^n}\overset{Euler}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{\left(\frac{3pR^2}{2r}\right)^n}=\frac{3}{2}\left(\frac{r}{pR^2}\right)^n=Md\ .$$

Aplicația 82

In ΔABC

$$\frac{a}{(b+\lambda c)^{n+1}r_a^n}+\frac{b}{(c+\lambda a)^{n+1}r_b^n}+\frac{c}{(a+\lambda b)^{n+1}r_c^n}\geq\frac{3}{(\lambda+1)^n}\left(\frac{2r}{pR^2}\right)^n,\lambda\geq 0,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+\lambda c)^{n+1}r_a^n}=\sum\frac{\frac{a^{n+1}}{(b+\lambda c)^{n+1}}}{a^nr_a^n}=\sum\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(ar_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum ar_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{\left(2p(2R-r)\right)^n}\overset{Euler}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{\left(\frac{3pR^2}{2r}\right)^n}=\frac{3}{(\lambda+1)^n}\left(\frac{2r}{pR^2}\right)^n=Md\ .$$

Aplicația 83

In ΔABC

$$\frac{a}{(b+c)^{n+1}m_a^n}+\frac{b}{(c+a)^{n+1}m_b^n}+\frac{c}{(a+b)^{n+1}m_c^n}\geq\frac{3}{2}\left(\frac{1}{2pR}\right)^n,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+c)^{n+1}m_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)^{n+1}}}{a^nm_a^n}=\sum\frac{\left(\frac{a}{b+c}\right)^{n+1}}{(am_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{\left(\sum am_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}\overset{Euler}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}=\frac{3}{2}\left(\frac{1}{2pR}\right)^n=Md$$. Am folosit mai sus $\sum am_a\leq 3Rp$.

Aplicația 84

In ΔABC

$$\frac{a}{(b+\lambda c)^{n+1}m_a^n}+\frac{b}{(c+\lambda a)^{n+1}m_b^n}+\frac{c}{(a+\lambda b)^{n+1}m_c^n}\geq\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n,\lambda\geq 0,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+\lambda c)^{n+1}m_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)\lambda^{n+1}}}{a^n m_a^n}=\sum\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(am_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum am_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{(3Rp)^n}=\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n=Md$$. Am folosit mai sus $\sum am_a\leq 3Rp$.

Aplicația 85

In ΔABC

$$\frac{a}{(b+c)^{n+1}w_a^n}+\frac{b}{(c+a)^{n+1}w_b^n}+\frac{c}{(a+b)^{n+1}w_c^n}\geq\frac{3}{2}\left(\frac{1}{2pR}\right)^n,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+c)^{n+1}w_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)^{n+1}}}{a^n w_a^n}=\sum\frac{\left(\frac{a}{b+c}\right)^{n+1}}{(aw_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{\left(\sum aw_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}\overset{Euler}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}=\frac{3}{2}\left(\frac{1}{2pR}\right)^n=Md$$. Am folosit mai sus $\sum aw_a\leq 3Rp$.

Aplicația 86

In ΔABC

$$\frac{a}{(b+\lambda c)^{n+1}w_a^n}+\frac{b}{(c+\lambda a)^{n+1}w_b^n}+\frac{c}{(a+\lambda b)^{n+1}w_c^n}\geq\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n,\lambda\geq 0,n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+\lambda c)^{n+1}w_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)\lambda^{n+1}}}{a^n w_a^n}=\sum\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(aw_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum aw_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{(3Rp)^n}=\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n=Md$$. Am folosit mai sus $\sum aw_a\leq 3Rp$.

Aplicația 87

In ΔABC

$$\frac{a}{(b+c)^{n+1}s_a^n}+\frac{b}{(c+a)^{n+1}s_b^n}+\frac{c}{(a+b)^{n+1}s_c^n}\geq\frac{3}{2}\left(\frac{1}{2pR}\right)^n, n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+c)^{n+1}s_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)^{n+1}}}{a^n s_a^n}=\sum\frac{\left(\frac{a}{b+c}\right)^{n+1}}{(as_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{\left(\sum as_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}\overset{Euler}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{(3Rp)^n}=\frac{3}{2}\left(\frac{1}{2pR}\right)^n=Md$$

. Am folosit mai sus $\sum as_a\leq\sum am_a\leq 3Rp$.

Aplicația 88

In ΔABC

$$\frac{a}{(b+\lambda c)^{n+1}s_a^n}+\frac{b}{(c+\lambda a)^{n+1}s_b^n}+\frac{c}{(a+\lambda b)^{n+1}s_c^n}\geq\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n, \lambda\geq 0, n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{(b+\lambda c)^{n+1}s_a^n}=\sum\frac{\frac{a^{n+1}}{(b+c)\lambda^{n+1}}}{a^n s_a^n}=\sum\frac{\left(\frac{a}{b+\lambda c}\right)^{n+1}}{(as_a)^n}\overset{Radon}{\geq}\frac{\left(\sum\frac{a}{b+\lambda c}\right)^{n+1}}{\left(\sum as_a\right)^n}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{(3Rp)^n}=\frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{pR}\right)^n=Md$$

. Am folosit mai sus $\sum as_a\leq\sum am_a\leq 3Rp$.

Remarca.

La inegalitățile de mai sus egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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2. A Result by Using Beta Function

By Toyesh Prakash Sharma

Abstract

In this article, we are introducing a result by using beta function. Here we provide our result on behalf of observations we will claim our result right.

Main Result

With reference to [1] we have,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Put m=1-n as a result

$$\Gamma(1-n)\Gamma(n) = \int_0^\infty \frac{x^{n-1}}{1+x}dx$$

Now, we can rewrite above equation as

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\pi}{\sin(n\pi)} \ , \ n \in]0,1[$$

It was originally discovered by Euler [2], using above equation and by using DUIS (Differentiation Under Integral Sign) [3] Article on Brilliant " Differentiation Under Integral Sign " https://brilliant.org/wiki/differentiate-through-the-integral/

We get,

$$\partial_n \int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \partial_n \frac{\pi}{\sin(n\pi)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)}log_e(x)dx = -\frac{\pi^2}{\sin^2(n\pi)}\cos(n\pi)$$

Consider it as equation (1). Then, now using eq. (1) and DUIS we can write that

$$\partial_n \int_0^\infty \frac{x^{n-1}}{(1+x)}log_e(x)dx = -\pi^2\partial_n\left[\frac{\cos(n\pi)}{\sin^2(n\pi)}\right]$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)}log_e^2(x)dx = \frac{\pi^3}{\sin^3(n\pi)}[1+\cos^2(n\pi)]$$

Consider it as equation (II). Then, now using eq. (ii) and DUIS we can write that

$$\partial_n \int_0^\infty \frac{x^{n-1}}{(1+x)}log_e^2(x)dx = \pi^3\partial_n\left[\frac{1+\cos^2(n\pi)}{\sin^3(n\pi)}\right]$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)} log_e^3(x)dx = -\frac{\pi^4 \cos(n\pi)}{\sin^4(n\pi)}[5+\cos^2(n\pi)]$$

Consider it as equation (iii). Then, now using eq. (iii) and DUIS we can write that

$$\partial_n \int_0^\infty \frac{x^{n-1}}{(1+x)} log_e^3(x)dx = -\pi^4 \partial_n \left[\frac{5\cos(n\pi)+\cos^3(n\pi)}{\sin^4(n\pi)}\right]$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)} log_e^4(x)dx = \frac{\pi^5}{\sin^5(n\pi)}[5+(15+\pi)\cos^2(n\pi)+\cos^4(n\pi)]$$

Consider it as equation (iv). And this will go on...

Now we can take some observations from eq. (I) and eq. (II). So, our eq. (I) and eq. (ii) are

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)} log_e(x)dx = -\frac{\pi^2}{\sin^2(n\pi)}\cos(n\pi)$$

And

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)} log_e^3(x)dx = -\frac{\pi^4 \cos(n\pi)}{\sin^4(n\pi)}[5+\cos^2(n\pi)]$$

As we can observe the pattern that for every odd number as power of $\ln^m x$ we get cosine term common on the right-hand side.

If we supposed to calculate the value of integral contain odd powered log there in integrals of such kind when above used n=1/2.

$$\int_0^\infty \frac{log_e(x)}{(1+x)\sqrt{x}}dx = \left(-\frac{\pi^2}{\sin^2(n\pi)}\cos(n\pi)\right)_{n=\frac{1}{2}}$$

$$\Rightarrow \int_0^\infty \frac{log_e(x)}{(1+x)\sqrt{x}}dx = 0$$

Likewise

$$\int_0^\infty \frac{x^{n-1}}{(1+x)} log_e^3(x)dx = \left(-\frac{\pi^4 \cos(n\pi)}{\sin^4(n\pi)}[5+\cos^2(n\pi)]\right)_{n=\frac{1}{2}} = 0$$

Then, in similar fashion we obtain

$$\int_0^\infty \frac{log_e^m(x)}{(1+x)\sqrt{x}}dx = 0.$$

For m belongs to the odd numbers.

Theorem 1. *If n>0 and m belong to odd number. Then,*

$$\int_0^\infty \frac{x^{n-1}\cdot \ln^m x}{(1+x)\cdot\sqrt{x}}dx = 0$$

Proof: we can say that for integrals of above kind if we assume m=5,7,9 or other even numbers then we found cos(n π) in all of them and we also knows for $\frac{2n-1}{2}\pi = n\pi - \frac{\pi}{2}, n > 0 \cos\left(n\pi - \frac{\pi}{2}\right) = 0$. Accordingly on behalf of above expressed equations we can write the generalization equation of all i.e.

$$\int_0^\infty \frac{x^{n-1}}{(1+x)} \cdot \ln^m x \, dx = 0$$

With replacing n with $\frac{2n-1}{2}$ because when it consists, we got right hand side reduced to zero due to presented con(nx) there. With it we obtained

$$\int_0^\infty \frac{x^{\left(\frac{2n-1}{2}-1\right)}}{(1+x)} \cdot \ln^m x \, dx = 0$$

$$\int_0^\infty \frac{x^{n-1} \cdot \ln^m x}{(1+x)\cdot\sqrt{x}} dx = 0$$

Hence Proof.

Conclusion

In conclusion we can say that, in this article we have showed a result that gives help us to save our precious time to solve it from start, first judge the integral if it comes under the range of integrals or here, mentioned theorem then you directly say its answer is zero only. Many times, we found such a situation where we can apply, well known integrals on place of proving them likewise we may use them while dealing with some other types of integrals.

Acknowledgement

Both the authors are thankful to Prof. Neculai Stanciu and Editor.

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3. Trei soluţii ale unei probleme din RMT2/2020

Daniel Văcaru, Colegiul Economic „*Maria Teiuleanu*", Piteşti

Enunţul problemei este:

VII.525. Aflați aria trapezului *ABCD* ştiind că $AB \| CD$, *AB* = 3 cm, *BC* = 4 cm, *AD* = 5 cm, *CD* = 6 cm.

dr.RMT

Prima soluţie (discutată cu elevii **Mihai Lăcrămioara, Stefan Dennis, Deaconeasa Ştefana, Isbăşescu Cristina**, clasa a X – a) ar consta în a completa trapezul până la triunghiul *ECD*, unde $\{E\}=AD\cap BC$. Observăm că *AB* este linie mijlocie în triunghiul *ECD*, de unde $EC=2\cdot BC=8\,cm$, iar $ED=2\cdot AD=10\,cm$. Să mai observăm şi că triunghiul *ECD* este dreptunghic, cu ipotenuza *ED*, de unde aria cerută este $\frac{3A_{ECD}}{4}=\frac{3\,EC\cdot CD}{8}=18\,cm^2$.

În cea de – a **doua soluţie** discutată cu eleva **Bebu Elena** (clasa a IX – a), notăm $E=pr_{CD}B$ şi $F=pr_{CD}A$ şi fie $FC \overset{not}{=} x$. Evident, $EF=3\,cm$, deoarece *ABEF* este dreptughi. Atunci aplicăm **teorema lui Pitagora** pentru a exprima $AF=BE$ şi găsim

$$AF^2=AD^2-DF^2=5^2-(3-x)^2(1)$$

şi $BE^2=BC^2-CF^2=4^2-x^2(2)$. Egalând, obţinem

$$5^2-(3-x)^2=4^2-x^2\Rightarrow 5^2-4^2=(3-x)^2-x^2\Rightarrow(5-4)\cdot(5+4)=[(3-x)-x]\cdot[(3-x)+x]\Rightarrow 9=3\cdot(3-2x)$$
$$\Rightarrow 3-2x=3\Rightarrow x=0$$

arătând că $AF\perp CD$, deci trapezul *ABCD* este dreptunghic. Atunci aria sa este

$$A_{ABCD}=\frac{(AB+CD)\cdot AD}{2}=\frac{(3+6)\cdot 4}{2}=18\,cm^2$$

Cea de – a **treia soluţie** constă în a considera punctul *X*, mijlocul lui [*CD*], şi a observa că patrulaterul *ABCX* este un paralelogram, căci $AB=CX=3\,cm$ şi, prin ipoteză $AB\|CD=CX$. Să observăm că triunghiul *ADX* are laturile *AD* = 5 cm, *DX* = 3 cm şi *AX* = 4cm, este, deci, **dreptunghic** (**reciproca teoremei lui Pitagora**) şi *ABCX* este **dreptunghi**. Atunci, cu **proprietatea de aditivitate a ariei**, avem

$$A_{ABCD}=A_{ABCX}+A_{ADX}=3\cdot 4+\frac{3\cdot 4}{2}=18\,cm^2$$

4.Teorema chinezească a resturilor și exemple

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Teorema chinezească a resturilor este un rezultat provenit din teoria numerelor, cu aplicații în criptografie. Teorema a fost cunoscută de matematicienii chinezi din secolul al III-lea, apărând într-o carte a matematicianului Sun Tzu, iar apoi, în 1247, într-o altă carte a lui Qin Jiushao.

Lema chineză a resturilor

Dacă $\boldsymbol{m_1, m_2, \dots, m_n \in Z}$ cu $\boldsymbol{(m_i, m_j) = 1}$, pentru orice $\boldsymbol{i, j \in \{1, 2, \dots, n\}}$, $\boldsymbol{i \neq j}$ și dacă $\boldsymbol{a_1, a_2, \dots, a_n \in Z}$, atunci există un număr întreg $\boldsymbol{x}$, soluție a sistemului de congruențe

$$\begin{cases} \boldsymbol{x \equiv a_1 \ mod \, m_1} \\ \boldsymbol{x \equiv a_2 \ mod \, m_2} \\ \dots \dots \dots \dots \dots \dots \dots \\ \boldsymbol{x \equiv a_n \ mod \, m_n} \end{cases}$$

Pentru orice altă soluție $\boldsymbol{y}$ a sistemului, avem $\boldsymbol{y \equiv x \, mod \, m}$, unde $\boldsymbol{m = m_1 \cdot m_2 \cdot \dots \cdot m_n}$.

Demonstrație.

Fie $k_i = \frac{m}{m_i} \in N$ pentru oricare $i \in \{1,2, \dots, n\}$. Avem că $(m_i, k_i) = 1$ și atunci există $u_i, v_i \in Z$ astfel încât $m_i \cdot u_i + k_i \cdot v_i = 1$. Notăm $k_i \cdot v_i = c_i$ și obținem $m_i \cdot u_i + c_i = 1$, de unde rezultă că $c_i \equiv 1 \, mod \, m_i$, unde $i \in \{1,2, \dots, n\}$ și $c_i \equiv 0 \, mod \, m_j$ pentru $i \neq j$.

Din $m_i \cdot u_i + k_i \cdot v_i = 1$ rezultă că $k_i \cdot v_i \equiv 1 \, mod \, m_i$ adică $v_i = k_{i \, mod \, m_i}^{-1}$ (inversul multiplicativ al lui k_i $modulo \, m_i$.

Construim pe x de forma

$$x = \sum_{i=1}^{n} a_i c_i$$

de unde putem scrie

$$\boldsymbol{x = \sum_{i=1}^{n} a_i \cdot k_i \cdot k_{i \, mod \, m_i}^{-1}}$$

Atunci $x \equiv b_i \, mod \, m_i$ pentru orice $i \in \{1,2, \dots, n\}$.

Dacă y este altă soluție a sistemului, avem $x \equiv y \, mod \, m_i$, adică $m_i \,|\, x - y$ pentru $i \in \{1,2,\dots,n\}$. Cum $(m_i, m_j) = 1$ pentru orice $i \neq j$, rezultă că $m_1 \cdot m_2 \cdot \dots \cdot m_n \,|\, x - y$, adică $m \,|\, x - y$. De aici avem $y \equiv x \, mod \, m$. Mulțimea tuturir soluțiilor este $S = \{x + km \,|\, k \in Z\}$

Caz particular ($n = 2$ **)**

Dacă $\begin{cases} x \equiv a \, mod \, p \\ x \equiv b \, mod \, q \end{cases}$, unde $(p, q) = 1$ atunci

$$x \equiv (a \cdot q \cdot q^{-1}_{mod\, p} + b \cdot p \cdot p^{-1}_{mod\, q}) \, mod \, pq$$

Exercițiu

Să se determine numerele întregi care dau restul $\mathbf{7}$ prin împărțire la $\mathbf{43}$ și restul $\mathbf{3}$ prin împărțire la $\mathbf{47}$ **.**

Rezolvare

Din enunț, deducem că $\begin{cases} x \equiv 7 \, mod \, 43 \\ x \equiv 3 \, mod \, 47 \end{cases}$, cum $(43, 47) = 1$ atunci

$$x \equiv (7 \cdot 47 \cdot 47^{-1}_{mod\, 43} + 3 \cdot 43 \cdot 43^{-1}_{mod\, 47}) \, mod \, 43 \cdot 47$$

$47^{-1} \, mod \, 43 = 11$ deoarece $47 \cdot 11 = 43 \cdot 12 + 1$

$43^{-1} \, mod \, 47 = 35$ deoarece $43 \cdot 35 = 47 \cdot 32 + 1$

Obținem $x \equiv (7 \cdot 47 \cdot 11 + 3 \cdot 43 \cdot 35) \, mod \, 2021$

$$x \equiv (3691 + 4515) \, mod \, 2021$$

$$x \equiv (1598 + 473) \, mod \, 2021$$

$$x \equiv 2071 \, mod \, 2021$$

$$x \equiv 50 \, mod \, 2021$$

De aici rezultă că $x \in \{50 + 2021k \,|\, k \in Z\}$

Caz particular ($n = 3$ **)**

Dacă $\begin{cases} x \equiv a \, mod \, p \\ x \equiv b \, mod \, q \\ x \equiv c \, mod \, r \end{cases}$, unde p, q, r sunt numere întregi prime între ele două câte două, atunci

$$x \equiv \left(a \cdot (qr) \cdot (qr)^{-1}_{mod\, p} + b \cdot (pr) \cdot (pr)^{-1}_{mod\, q} + c \cdot (pq) \cdot (pq)^{-1}_{mod\, r}\right) mod \, pqr$$

Exercițiu

Să se determine numerele întregi care, prin împărțire la **2**, **3**, **337**, dau resturile **1**, **2** respectiv **7** **.**

Rezolvare

Din teorema împărțirii cu rest, rezultă sistemul de congruențe $\begin{cases} x \equiv 1\ mod\ 2 \\ x \equiv 2\ mod\ 3 \\ x \equiv 7\ mod\ 337 \end{cases}$

Cum $2, 3, 337$ sunt numere întregi prime între ele două câte două, atunci

$$x \equiv \left(1 \cdot (3 \cdot 337) \cdot (3 \cdot 337)^{-1}_{mod\ 2} + 2 \cdot (2 \cdot 337) \cdot (2 \cdot 337)^{-1}_{mod\ 3} + 7 \cdot (2 \cdot 3) \cdot (2 \cdot 3)^{-1}_{mod\ 337}\right)\ mod\ 2 \cdot 3 \cdot 337$$

$$x \equiv \left(1 \cdot 1011 \cdot 1011^{-1}_{mod\ 2} + 2 \cdot 674 \cdot 674^{-1}_{mod\ 3} + 7 \cdot 6 \cdot 6^{-1}_{mod\ 337}\right)\ mod\ 2022$$

Cum $1011^{-1}\ mod\ 2 = 1^{-1}\ mod\ 2 = 1$

$674^{-1}\ mod\ 3 = 2^{-1}\ mod\ 3 = 2$

$6^{-1}\ mod\ 337 = 281$ deoarece $6 \cdot 281 = 337 \cdot 5 + 1$

Se obține

$$x \equiv (1 \cdot 1011 \cdot 1 + 2 \cdot 674 \cdot 2 + 7 \cdot 6 \cdot 281)\ mod\ 2022$$

$$x \equiv (1011 + 2696 + 11\,802)\ mod\ 2022$$

$$x \equiv (1011 + 674 + 1692)\ mod\ 2022$$

$x \equiv (1685 + 1692)\ mod\ 2022$

$x \equiv 1355\ mod\ 2022$

Rezultă că $x \in \{1355 + 2022k \mid k \in Z\}$

Bibliografie

https://ro.wikipedia.org/wiki/Teorema_chinezeasc%C4%83_a_resturilor

5. PROBLEME INTERESANTE

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Clasa a V-a

1) Cu 3 cifre de 5 şi o cifră 1 şi cu operaţiile învăţate obţineţi 24.

Soluţie.

Soluţia 1: $5\cdot 5-1^5=24$

Soluţia 2: $5\cdot\left(5-\frac{1}{5}\right)=24$

Clasa a VI-a

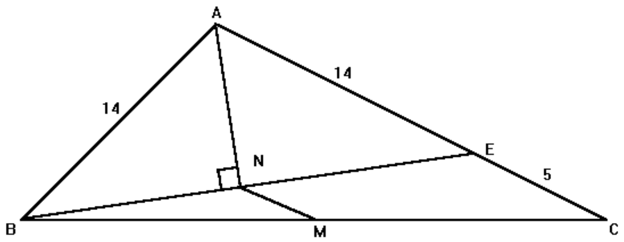
1) Găsiţi cel mai mic număr natural a cărui ultimă cifră este 6, astfel încât dacă ultima cifră este mutată în faţa numărului(exemplu: 14576 prin mutare devine 61457) acesta se măreşte de 4 ori.

Soluţie.

Avem $\overline{a_1a_2....a_m6}$ numărul dat. Notăm $n=\overline{a_1a_2....a_m}$ (m cifre) deci numărul dat devine $\overline{n6}$, iar după mutare $\overline{6n}$. Condiţia din enunţ devine $4(10n+6)=6\cdot 10^m+n$, prelucrată obţinem $13n+8=2\cdot 10^m$ deci n este număr par şi are forma $n=2k$. Înlocuind obţinem $13k=10^m-4$. Cele mai mici valori care verifică relaţia sunt $m=5,\ k=7692$. Numărul cerut este 153846.

2) În triunghiul ABC, M este mijlocul laturii BC, AN este bisectoarea unghiului BAC, BN este perpendiculară pe AN. Dacă $AB=14$, $AC=19$ să se afle MN.

Soluţie.

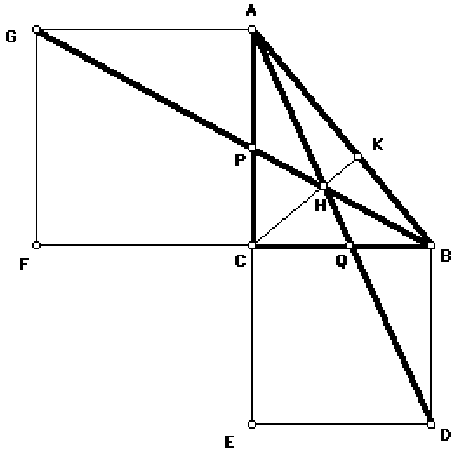


Prelungim BN şi notăm $BN\cap AC=\{E\}$. Se arată uşor că $\Delta BNA\equiv\Delta ENA$, deci N este mijlocul segmentului BE; avem $AB=AE=14$ şi $EC=5$. MN este deci linie mijlocie în triunghiul BEC $\rightarrow MN=\frac{EC}{2}=\frac{5}{2}$.

Clasa a VII-a

1) Pe catetele triunghiului dreptunghic ABC, $m(\sphericalangle C)=90^0$, se construiesc în exterior, pătratele ACFG şi BCED. Notăm $\{H\}=AD\cap BG$, $\{K\}=CH\cap AB$. Arătaţi că:
a) $\frac{AP}{CP}=\frac{CQ}{BQ}=\frac{AC}{BC}$ b) $\frac{1}{PC}=\frac{1}{AC}+\frac{1}{BC}$ şi $\frac{1}{QC}=\frac{1}{AC}+\frac{1}{BC}$

Soluţie.



a) Din $\Delta PAG\sim\Delta PCB\rightarrow\frac{PA}{PC}=\frac{AG}{CB}=\frac{PG}{PB}$; Din $\Delta QCA\sim\Delta QBD\rightarrow\frac{QC}{QB}=\frac{CA}{BD}=\frac{QA}{QD}$

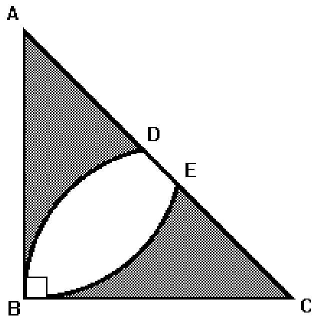
Avem şi $AG=AC,\ BD=BC$ de unde rezultă $\frac{AP}{CP}=\frac{CQ}{BQ}=\frac{AC}{BC}$.

b) Din $\Delta GFB\sim\Delta PCB\rightarrow\frac{GF}{PC}=\frac{FB}{CB}=\frac{GB}{PB}$ şi $\Delta AED\sim\Delta ACQ\rightarrow\frac{AE}{AC}=\frac{ED}{CQ}=\frac{AD}{AQ}$

Scriem $GF=AC$, $FB=FC+CB\rightarrow\frac{AC}{PC}=\frac{AC+BC}{BC}$ de unde $\frac{1}{PC}=\frac{1}{AC}+\frac{1}{BC}$. Analog se arată relaţia cealaltă.

2) Se dă triunghiul dreptunghic şi isoscel ABC cu $m(\sphericalangle B)=90^0$ şi $BC=AB=2\ cm$. Un arc de cerc de rază 2 cm cu centrul în C intersectează ipotenuza în D, iar un arc de cerc de rază 2 cm cu centrul în A intersectează ipotenuza în E, conform figurii. Aflaţi aria porţiunii haşurate.

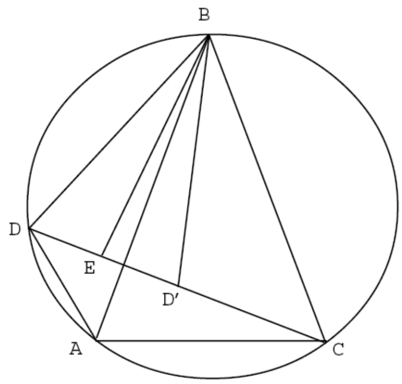
Soluţie.



Se calculează aria unui sector de disc $A_{sector} = \frac{1}{8}\pi R^2 = \frac{\pi}{2}\ cm^2$; se calculează aria triunghiului $A_\Delta = \frac{AB \cdot BC}{2} = \frac{4}{2} = 2\ cm^2$; se calculează aria unei porţiuni haşurate prin diferenţă $A_h = 2 - \frac{\pi}{2} = \frac{4-\pi}{2}$ şi aceasta se înmulţeşte cu 2 pentru aflarea întregii porţiuni haşurate: $A_{final} = (4-\pi)\ cm^2$.

3) Pe cercul de centru O se iau punctele A, B, C astfel încât B să fie mijlocul arcului AC. Se alege un punct D pe arcul mic AB, şi fie E piciorul perpendicularei din B pe DC. Arătaţi că $CE = ED + DA$.

Soluţie.



Fie D' simetricul punctului D faţă de dreapta BE. Arătăm că $\Delta BDA \equiv \Delta BD'C$ (folosind cazul L.U.L.). Din simetria construită avem $BD = BD'$; avem $BA = BC$ (la arce congruente corespund coarde congruente).Folosind faptul că patrulaterul ACBD este înscris în cerc:avem $\sphericalangle BDD' \equiv \sphericalangle BD'D$ (din simetrie), $\sphericalangle BAC \equiv \sphericalangle BCA$ (triunghiul BAC isoscel), $\sphericalangle BDD' \equiv \sphericalangle BAC$ (patrulaterul este înscris în cerc). Din Toate acestea rezultă că şi $\sphericalangle DBD' \equiv \sphericalangle BAC$ care au o parte comună şi anume $\sphericalangle ABD'$. Deci rezultă că $\sphericalangle DBA \equiv \sphericalangle D'BC$. Obţinem astfel congruenţa celor două triunghiuri, din care avem $DA = D'C$. În final $CE = CD' + D'E = AD + ED$.

Clasa a VIII-a

1) Pentru m şi n numere întregi definim o nouă operaţie notată „* ” astfel: $m * n = \frac{m+n}{mn+4}$. Calculaţi $((((.....(2007*2006)*2005)*2004)........)*1)*0)$

Soluţie.

Observăm că pentru $n=2$ avem $m*2=\frac{m+2}{2m+4}=\frac{m+2}{2(m+2)}=\frac{1}{2}$

Notând cu $x=((((.....(2007*2006)*2005)*2004)........)*3)$ ceea ce aveam de calculat devine $(((x*2)*1)*0)=((\frac{1}{2}*1)*0)=\frac{1}{3}*0=\frac{1}{12}$.

Clasa a IX-a

1) Fie a, b, c numere reale pozitive. Găsiţi cel mai mare număr real C, pentru care este satisfăcută inegalitatea: $\frac{(2a+b+c)(a+2b+c)(a+b+2c)}{abc}\ge C$.

Soluţie.

Din inegalitatea mediilor avem $a+a+b+c\ge 4\sqrt[4]{a^2bc}$, $a+b+b+c\ge 4\sqrt[4]{ab^2c}$, $a+b+c+c\ge 4\sqrt[4]{abc^2}$. Înmulţind realţiile membru cu membru obţinem

$$\frac{(2a+b+c)(a+2b+c)(a+b+2c)}{abc}\ge 64\to C\ge 64$$

Punând $a=b=c$ în relaţia anterioară, obţinem $64\ge C$ deci $C=64$.

2) Arătaţi că ecuaţia $2x^7=1-x$ are o singură soluţie în intervalul $[0;1]$.

Soluţie.

Dacă $x>1$ atunci $1-x<0$ şi $2x^7>1$.

Dacă $x<0$ atunci $1-x>0$ şi $2x^7<0$. Deci soluţia este în intervalul $[0;1]$. Arătăm unicitatea. Presupunem că există 2 soluţii distincte x_1, x_2 astfel încât $2x_1^7+x_1-1=2x_2^7+x_2-1\to 2(x_1^7-x_2^7)+(x_1-x_2)=0$. Cum am presupus că soluţiile sunt distincte avem $2\left(\sum_{i=0}^{6}x_1^i x_2^{6-i}\right)+1=0$. Cum $x_1\ge 0,\ x_2\ge 0$ ultima relaţie devine imposibilă(membrul stâng al ei este mai mare sau egal cu 1). Condradicţie, deci avem unicitatea soluţiei.

Clasa a X-a

1) Ştiind că $a^{\log_b c}+b^{\log_c a}=m$, găsiţi valoarea expresiei $c^{\log_b a}+a^{\log_c b}$.

Soluţie.

$$a^{\log_b c}+b^{\log_c a}=a^{\frac{\log_a c}{\log_a b}}+b^{\frac{\log_b a}{\log_b c}}=c^{\frac{1}{\log_a b}}+a^{\frac{1}{\log_b c}}=c^{\log_b a}+a^{\log_c b}=m$$