Varianta 035

Subiectul I

a) 8i. b) 1. c) $4\sqrt{3}$. d) 4,8. e) x-y+z = 4. f) x+y = 2 .

Subiectul II

1. a) 4. b) $\hat{2}$. c) $\frac{9}{2}$. d) $A_4^3 = 24$. e) $\frac{4}{5}$.

2. a) 0. b) $(1-x)e^{-x}$. c) $e^x \geq ex$. d) x = 2, singurul punct de inflexiune. e) $\frac{e-2}{e}$.

Subiectul III

a) Avem $I_3 \cdot X = X \cdot I_3$, $(\forall)\ X \in M_3(\mathbf{C})$, deci $I_3 \in S$.

b) Toti minorii de ordinal doi ai matricelor E_i, $i \in \{1,2,3\}$ sunt nuli ,deci rang(E_i) = 1, $\forall i \in \{1,2,3\}$.

c) Fie A = $\begin{pmatrix} a_1 a_2 a_3 \\ b_1 b_2 b_3 \\ c_1 c_2 c_3 \end{pmatrix} \in M_3(\mathbf{C})$ .Din $A \cdot E_i = E_i \cdot A$, $(\forall) i \in \{1,2,3\}$ obtinem :

$a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$ si $a_1 = b_2 = c_3$ .Deducem ca exista $a \in \mathbf{C}$ astfel incat

$A = aI_3$.

d) Fie $A \in S$. Rezulta ca $A \cdot X = X \cdot A$, $(\forall) X \in M_3(\mathbf{C})$. In particular $A \cdot E_i = E_i \cdot A$, $(\forall) i \in \{1,2,3\}$. Din c) $\Rightarrow \exists\ a \in \mathbf{C}$ astfel incat A = a I_3 ,deci $\mathbf{S} \subset \{aI_3 | a \in \mathbf{C}\}$.

Cum $(aI_3)X = X(aI_3)$, $(\forall) X \in M_3(\mathbf{C})$ avem $\{aI_3 | a \in \mathbf{C}\} \subset \mathbf{S}$. Asadar $\mathbf{S} = \{aI_3 | a \in \mathbf{C}\}$.

e) Se verifica axiomele inelului .

f) Fie $a, b \in \mathbf{C}$ arbitrar alese asfel incat f(a) = f(b). Obtinem $aI_3 = bI_3$, deci a = b. Deducem ca functia f este injectiva . Surjectivitatea functiei f este evidenta.

g) Presupunem ca functia g este bijectivă $\Rightarrow \exists\ X, Y \in \mathbf{M}_3(\mathbf{R})$ astfel incat g(X) = O_3 si g(Y) = I_3. Alegem A = X si B = O_3 $\Rightarrow$ g(O_3) = O_3. Alegem A = Y si B = I_3 $\Rightarrow$ g(I_3) = I_3. Fie $M = \{A \in \mathbf{M}_3(\mathbf{R}) | AX = XA, (\forall) X \in \mathbf{M}_3(\mathbf{R})\}$. Aratam ca g(M) = S.
Daca $A \in \mathbf{M}$ avem $AX = XA$. Rezulta ca $g(A) \cdot g(X) = g(X) \cdot g(A)$ si cum g este bijectiva $\Rightarrow$ $\Rightarrow g(A) \in \mathbf{S}$. Din $B \in \mathbf{S} \Rightarrow \exists A \in \mathbf{M}_3(\mathbf{R})$ astfel incat $g(A) = B$. Avem $g(A) \cdot Y = g(A) \cdot g(X)$ $= Y \cdot g(A) = g(X) \cdot g(A)$ sau g(AX) = g(XA) sau AX = XA, deci $A \in \mathbf{M}$.

Fie functia f:**R**→**C**, f(b) = a daca $g(bI_n) = aI_n$. Functia f este bijectiva si avem f(xy) = f(x)f(y). Fie $x \in \mathbf{R}$ cu proprietatea f(x) = i . Atunci $f(x^4) = i^4 = 1 = f(1)$, deci $x^4 = 1$ sau $x \in \{-1,1\}$. Dar f(1) = 1, iar f(-1) = -1, deci f(x) ≠ i , fals .

Subiectul IV

a) Avem $f_1(x) = \int_0^x (t - \sin t)dt = \frac{x^2}{2} + \cos x - 1, (\forall) x \in \mathbf{R}$.

b) $f_1^{'}(x) = x - \sin x, (\forall) x \in \mathbf{R}$ si $f_1^{''}(x) = 1 - \cos x \geq 0, (\forall) x \in \mathbf{R}$. .Deducem ca functia f_1 este convexa pe **R**.

c) vezi varianta 45, subiectul IV, e).

d) vezi varianta 45, subiectul IV, f).

e) Din punctual d) $\Rightarrow$ $f_{4n-1}(x) > 0, (\forall) n \in \mathbf{N}^*, (\forall) x \in [0, \infty)$.

$$\text{De aici obtinem} \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .... + \frac{x^{4n}}{(4n)!}, (\forall) n \in \mathbf{N}^*, (\forall) x \in [0, \infty).$$ Analog se arată că $\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .... + \frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!}$, $(\forall) x \in [0, \infty)$.

f) vezi varianta 45, subiectul IV, g).

g) Presupunem ca, $\cos 1 \in \mathbf{Q}$ si deci exista $p \in \mathbf{Z}, q \in \mathbf{N}^*$ astfel incat $\cos 1 = \frac{p}{q}$. Din e) $\Rightarrow$

$$1 - \frac{1}{2!} + \frac{1}{4!} - .... + \frac{1}{(4n)!} - \frac{1}{(4n+2)!} < \cos 1 < 1 - \frac{1}{2!} + \frac{1}{4!} - ... + \frac{1}{(4n)!}, (\forall) n \in \mathbf{N}^*.$$

In particular $1 - \frac{1}{2!} + \frac{1}{4!} - ... + \frac{1}{(4q)!} - \frac{1}{(4q+2)!} < \frac{p}{q} < 1 - \frac{1}{2!} + \frac{1}{4!} - ... + \frac{1}{(4q)!}$ si deci

$$-\frac{1}{(4q+2)!} < \frac{p}{q} - 1 + \frac{1}{2!} - \frac{1}{4!} + ... - \frac{1}{(4q)!} < 0;$$

Deducem ca $-1 < (4q+2)!\left(\frac{p}{q} - 1 + \frac{1}{2!} - \frac{1}{4!} + ... - \frac{1}{(4q)!}\right) < 0$.

Cum $(4q+2)!\left(\frac{p}{q} - 1 + \frac{1}{2!} - \frac{1}{4!} + ... - \frac{1}{(4q)!}\right) \in \mathbf{Z}$ obtinem o contradictie.