Varianta 044

Subiectul I

a) 1. b) 10. c) 0. d) $A = 5$. e) $\vec{v} \cdot \vec{w} = 1$. f) $\frac{3}{\sqrt{5}}$.

Subiectul II

1. a) $\frac{1}{2} \cdot \frac{2}{3} \cdot ... \cdot \frac{10}{11} = \frac{1}{11}$. b) $2^3 = 8$. c) $2^{2x} \in \{-1, 4\}, 2^{2x} = 4 \Rightarrow x = 1$. d) $\hat{0}, \hat{1}$ verifica, $\hat{2}, \hat{3}$ nu verifica. e) $\log_2 n \geq \frac{n-1}{2}, \forall n \in \{1,2,3,4\} \Rightarrow P = 1$.

2. a) $\lim_{x \to \infty} f(x) = 0,\ y = 0$. b) $f'(x) = \frac{-2x}{(x^2+4)^2}$. c) $x^2 + 4 \geq 4, \frac{1}{x^2+4} \leq \frac{1}{4}$. d) $f'(1) = \frac{-2}{25}$.

e) $\int_0^2 f(t)dt = \frac{1}{2} \operatorname{arctg} \frac{x}{2} \Big|_0^2 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$.

Subiectul III

a) Avem $E^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. $E^3 = I^3$. Evident ca $E, E^2, E^3 \in M$.

b) Deoarece $E \cdot E^2 = E^2 \cdot E = I_3 \Rightarrow E$ inversabil ; $E^{-1} = E^2$.

c) Verificare. d) Calcul direct

e) Avem egalitatea: $a^2 + b^2 + c^2 - 3abc = \frac{1}{2}(a+b+c)[(b-c)^2 + (c-a)^2 + (a-b)^2]$.

Cum $a+b+c \geq 0 \Rightarrow (b-c)^2 + (c-a)^2 + (a-b)^2 \geq 0$, obtinem $a^3 + b^3 + c^3 - 3abc \geq 0$.

f) Fie $X = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in M_3(\mathbf{R})$, din XE= EX $\Rightarrow a_1 = b_2 = c_3;\ a_2 = b_3 = c_1;$

$a_3 = b_1 = c_2$, deci ($\exists$) a,b,c $\in \mathbf{R}$ a.i. $X = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \in M$.

g) Presupunem ca $(M(a,b,c))^n = M(a_n, b_n, c_n)$ si $a_n + b_n + c_n = (a+b+c)^n$

avem: $(M(a,b,c))^{1+n} =$

$$=(M(a,b,c))^n \cdot M(a,b,c) = \begin{pmatrix} a_n & b_n & c_n \\ c_n & a_n & b_n \\ b_n & c_n & a_n \end{pmatrix} \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} =$$

$$= \begin{pmatrix} a_n a + b_n c + c_n b & a_n b + b_n a + c_n c & a_n c + b_n b + c_n a \\ a_n c + b_n b + c_n a & a_n a + b_n c + c_n b & a_n b + b_n a + c_n c \\ a_n b + b_n a + c_n c & a_n c + b_n b + c_n a & a_n a + b_n c + c_n b \end{pmatrix} = M(a_{n+1}; b_{n+1}; c_{n+1})$$

Unde $\begin{aligned} a_{n+1} &= a_n a + b_n c + c_n b \\ b_{n+1} &= a_n b + b_n a + c_n c \\ c_{n+1} &= a_n c + b_n b + c_n a \end{aligned}$

Adunand relatiile obtinem

$a_{n+1} + b_{n+1} + c_{n+1} = (a_n + b_n + c_n)(a+b+c) = (a+b+c)^n \cdot (a+b+c) = (a+b+c)^{n+1}$.

h) Avem $XE = X \cdot X^{2007} = X^{2007} \cdot X = EX$ din f) $\Rightarrow X = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$. Conform punctului

g) Avem $X^{2007} = M(a_{2007}, b_{2007}, c_{2007})$, $a_{2007} + b_{2007} + c_{2007} = (a+b+c)^{2007}$, deci ecuatia devine $M(a_{2007}, b_{2007}, c_{2007}) = M(0,1,0) \Rightarrow a_{2007} = 0; b_{2007} = 1; c_{2007} = 0;$

Asadar $(a+b+c)^{2007} = 1 \Rightarrow \det X = 1$ deci $a^3 + b^3 + c^3 - 3abc = 1$ sau

$\frac{1}{2}((a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 1$ sau $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2$

a,b,c $\in \mathbf{Z} \Rightarrow a = b, b-c = \pm 1; c-a = \pm 1$ sau b=c. a-b=± 1. c-a =± 1 sau a=c , a-b= $\pm 1;$ c-a=$\pm 1;$ sau $a = c, a-b = \pm 1; c-a = \pm 1$ Se tine cont de a+b+c=1 si se obtin solutiile.

Subiectul IV

a) $I_0(X) = \ln(1+x)$. b) Inegalitatile sunt evidente $(1+t \geq 1)$.

c) Prin integrarea inegalitatilor de la punctul b) se obtine conditia.

d) Din c) cu criteriul clestelui se obtin $\lim_{n\to\infty} I_n(x) = 0$.

e) $I_n(x) + I_{n-1}(x) = \int_0^x \frac{t^{n-1}(t+1)}{1+t} dt = \frac{x^n}{n}, (\forall) x \in [0,1].$

f) Tinand cont de punctul e) avem :

$$\frac{x^n}{n} - \frac{x^{n-1}}{n-1} + \frac{x^{n-2}}{n-2} - \dots + (-1)^{n-1}\frac{x}{1} = (I_n(x) + I_{n-1}(x)) - (I_{n-1}(x) + I_{n-2}(x)) + \dots +$$
$$+ (-1)^{n-1}(I_1(n) + I_0(x)) = I_n(x) + (-1)^{n-1} I_0(x)$$

.

g) Din f) $\Rightarrow I_0(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n I_n(x), (\forall) x \in [0,1].$

Tinand cont de a) si d) se obtine concluzia.