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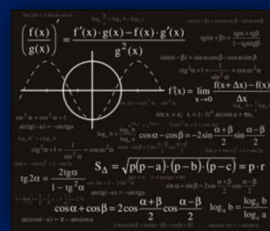

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REDACTORI PRINCIPALI ŞI SUSŢINĂTOR PERMANENŢI AI REVISTEI

NECULAI STANCIU, ROXANA MIHAELA STANCIU ŞI NELA CICEU

ARTICOLE REVISTĂ:

1. PROBLEMA LUNII APRILIE 2017

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Prove that in any triangle ABC with the lengths sides a,b,c the area S the semiperimeter p and circumradius R holds

$$\frac{9R}{8S}-\sum_{cyclic}\frac{1}{a+b}\geq\sum_{cyclic}\frac{(a-b)^2}{2(p-a)(p-b)(a+b+2c)}.$$

*Proposed by **Neculai Stanciu**, ''George Emil Palade'' School, Buzău, Romania and **Titu Zvonaru**, Comăneşti, Romania.*

*If you solve the problem, you can post your solution sending it by email to **revista@mateinfo.ro** . Deadline **29.04.2017***

2. PROBLEMA LUNII MARTIE 2017 - SOLUȚII

Prove that in all triangles ABC holds the inequality

$$\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}\geq 1.$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

***Solution*.** By Iranian's inequality, i.e. $(xy+yz+zx)\cdot\sum_{cyc}\frac{1}{(x+y)^2}\geq\frac{9}{4}, \forall x.y,z>0$ if we take $x=\sin A, y=\sin B, z=\sin C$, then

$$\left(\sum_{cyc}\sin A\sin B\right)\cdot\left(\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}\right)\geq\frac{9}{4}, (1).$$

We have the identity $\sum_{cyc}\sin A\sin B=\frac{s^2+4Rr+r^2}{4R^2}$, but by Euler's inequality we have $2r\leq R$ so $\sum_{cyc}\sin A\sin B\leq\frac{s^2+4R\cdot\frac{R}{2}+\frac{R^2}{4}}{4R^2}=\frac{4s^2+9R^2}{16R^2}$ and by Mitrinovic's inequality we have $s\leq\frac{3\sqrt{3}R}{2}$, so

$$\sum_{cyc}\sin A\sin B\leq\frac{4\cdot\frac{27R^2}{4}+9R^2}{16R^2}=\frac{36}{16}=\frac{9}{4}, (2).$$

Hence, by (1) and (2) yields the conclusion.

Other solutions to the problem

Solution 1 by ***Kevin Soto Palacios*** *– Huarmey – Peru*

Probar en un triángulo ABC, la siguiente desigualdad

$$\sum \frac{1}{(\sin A + \sin B)^2} \geq 1 \quad (A)$$

Siendo x, y, z números R^+, se cumple la siguiente desigualdad:

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$$

Realizamos las siguientes sustituciones

$$x = \sin A > 0, y = \sin B > 0, z = \sin C > 0$$

Luego $\forall\ x, y, z \in R$ se cumple lo siguiente:

$$xy + yz + zx \leq x^2 + y^2 + z^2 = \sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C \leq \frac{9}{4}$$

Finalmente la desigualdad es equivalente

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4(xy+yz+zx)} \geq 1 \quad (LQQD)$$

Solution 2 by ***Kevin Soto Palacios*** *– Huarmey – Peru*

Probar en un triángulo ABC, la siguiente desigualdad

$$\sum \frac{1}{(\sin A + \sin B)^2} \geq 1 \quad (A)$$

En un triángulo $ABC \rightarrow \sin A, \sin B, \sin C > 0$

Denotemos los siguientes cambios de variables

$$x = \sin A + \sin B > 0, y = \sin B + \sin C > 0, z = \sin C + \sin A > 0$$

Cumpliéndose que:

$$x + y + z \leq 3\sqrt{3} \wedge (x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9 \Leftrightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \sqrt{3}$$

Por la desigualdad de Cauchy en (A)

$$\sum\frac{1}{x^2}\geq\frac{1}{3}\left(\sum\frac{1}{x}\right)^2\geq\frac{1}{3}\left(\frac{9}{x+y+z}\right)^2\geq\frac{1}{3}\cdot 3=1\quad \textit{(LQQD)}$$

Solution 3 by ***Seyran Ibrahimov-Maasilli***-*Azerbaidian*

$$\sum\frac{1}{(\sin a+\sin b)^2}\geq\sum\frac{1}{2(\sin^2 a+\sin^2 b)}$$

$$a=2R\cdot\sin a,\qquad b=2R\cdot\sin b,\ \ c=2R\cdot\sin c$$

$$\frac{2R^2}{a^2+b^2}+\frac{2R^2}{a^2+c^2}+\frac{2R^2}{b^2+c^2}\overset{C-B-C}{\geq}\frac{2(R+R+R)^2}{2(a^2+b^2+c^2)}=\frac{9R^2}{a^2+b^2+c^2}\geq 1$$

$$\textit{because } 9R^2\geq a^2+b^2+c^2$$

Solution 4 by ***Myagmarsuren Yadamsuren***-*Darkhan-Mongolia*

$$\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}=\sum_{cyc}\frac{1^3}{(\sin A+\sin B)^2}\geq$$

$$\geq\frac{(1+1+1)^3}{\left(2\cdot(\sin A+\sin B+\sin C)\right)^2}=\frac{27}{(2\cdot\sum\sin A)^2}\geq\frac{27}{\left(2\cdot\frac{3\sqrt{3}}{2}\right)^2}=1$$

Solution 5 by ***Myagmarsuren Yadamsuren***-*Darkhan-Mongolia*

$$\sin A=\frac{a}{2R}\ \dots$$

$$\sum_{cyc}\frac{4R^2}{(a+b)^2}=4R^2\sum_{cyc}\frac{1}{a^2+2ab+b^2}=$$

$$=4R^2\cdot\sum_{cyc}\frac{1^2}{a^2+2ab+b^2}\geq 4R^2\cdot\left(\frac{(1+1+1)^2}{2(a^2+b^2+c^2+ab+bc+ca)}\right)$$

$$=18R^2\cdot\frac{1}{a^2+b^2+c^2+ab+bc+ca}\geq 1\quad \textit{(ASSURE)}$$

$$18R^2\geq a^2+b^2+c^2+ab+bc+ca$$

$$a^2+b^2+c^2=2p^2-8Rr-2r^2$$

$$ab+bc+ca=p^2+4Rr+r^2$$
$$p^2\le 4R^2+4Rr+3r^2$$
$$18R^2\ge 3p^2-4Rr-r^2$$
$$3p^2\le 18R^2+4Rr+r^2$$
$$3p^2\le 3(4R^2+4Rr+3r^2)\le 18R^2+4Rr+r^2$$
$$6R^2-8Rr-8r^2\ge 0$$
$$3R^2-4Rr-4r^2\ge 0$$
$$3R^2=2R^2+r^2\ge 4Rr+4r^2 \text{ (Euler)}$$

Solution 6 by **Soumitra Mandal**-*Chandar Nagore-India*

We know, $\frac{3}{2}\ge \sin A+\sin B+\sin C$, $\sin A=\frac{a}{2R}$, $\sin B=\frac{b}{2R}$ *and* $\sin C=\frac{c}{2R}$

Where a,b,c *are the sides of a triangle and* $R=$ *circum – radius*

$$9R^2\ge a^2+b^2+c^2\Rightarrow 9R^2\ge\frac{(a+b+c)^2}{3}\Rightarrow 27R^2\ge(a+b+c)^2$$

$$\therefore\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}=4R^2\sum_{cyc}\frac{1}{(a+b)^2}$$

$$\ge 4R^2\cdot\frac{27}{(a+b+b+c+c+a)^2}\left[\because\frac{1}{3}\sum_{cyc}\frac{1}{(a+b)^2}\ge\frac{9}{4}\cdot\frac{1}{(a+b+c)^2}\right]$$

$$=\frac{27R^2}{(a+b+c)^2}\ge 1 \text{ (proved) equality at } A=B=C=\frac{\pi}{3}$$

Solution 7 by **Soumava Chakraborty**-*Kolkata-India*

$$\sin A+\sin B=2\cos\frac{C}{2}\cos\frac{A-B}{2}$$
$$\because -\frac{\pi}{2}<\frac{A-B}{2}<\frac{\pi}{2},$$

$$\therefore 0 < \cos\frac{A-B}{2} \le 1$$

$$\Rightarrow 2\cos\frac{C}{2}\cos\frac{A-B}{2} \le 2\cos\frac{C}{2}$$

$$\Rightarrow \sin A + \sin B \le 2\cos\frac{C}{2} \Rightarrow (\sin A + \sin B)^2 \overset{(a)}{\le} 4\cos^2\frac{C}{2}$$

Similarly, $(\sin B + \sin C)^2 \overset{(b)}{\le} 4\cos^2\frac{A}{2}$ *and* $(\sin C + \sin A)^2 \overset{(c)}{\le} 4\sin^2\frac{B}{2}$

Using (a), (b), (c) $\sum\frac{1}{(\sin A+\sin B)^2} \ge \frac{1}{4}\sum\sec^2\frac{A}{2}$

Let $f(x) = \sec^2\left(\frac{x}{2}\right)\ \forall x \in (0,\pi)$

$$f''(x) = \sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right) + \frac{1}{2}\sec^4\left(\frac{x}{2}\right) > 0$$

$\therefore f(x)$ *is convex on* $(0,\pi)$

$$\therefore \sum\frac{1}{(\sin A+\sin B)^2} \ge \frac{1}{4}\sum\sec^2\frac{A}{2} \overset{Jensen}{\ge} \frac{1}{4}\cdot 3\sec^2\left(\frac{A+B+C}{6}\right)$$

$$= \frac{1}{4}\cdot 3\cdot\left(\frac{2}{\sqrt{3}}\right)^2 = 1 \quad \textit{(Proved)}$$

Solution 8 by ***Martin Lukarevski***-*Stip-Macedonia*

Solution. We use the well-known inequality $a^2+b^2+c^2 \leqslant 9R^2$. Since $ab+bc+ca \leqslant a^2+b^2+c^2$, the inequality

$$\sum(a+b)^2 \leqslant 36R^2$$

holds. By the law of sines the inequality (1) is equivalent to

$$\sum\frac{1}{(a+b)^2} \geqslant \frac{1}{4R^2}.$$

By Cauchy-Schwarz

$$\sum\frac{1}{(a+b)^2} \geqslant \frac{9}{\sum(a+b)^2} \geqslant \frac{1}{4R^2},$$

and we are done.

MARTIN LUKAREVSKI, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY "GOCE DELCEV" - STIP, MACEDONIA
E-mail address: martin.lukarevski@ugd.edu.mk

Solution 9 de ***Biro Istvan*** *– Romania*

Putem folosi următoarele relații cunoscute

(1) $\left(x^2+y^2\right)^2 \le 2\left(x^2+y^2\right), \quad \forall x,y\in\mathbb{R}$

(2) $\dfrac{3}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}} \le \dfrac{x+y+z}{3}, \quad \forall x,y,z\in\mathbb{R}_+^*$

(3) $\sin^2 A+\sin^2 B+\sin^2 C\le \dfrac{9}{4}$, *valabil în orice triunghi .*

Astfel obținem $\sum_{cyc}\dfrac{1}{(\sin A+\sin B)^2} \underset{(1)}{\ge} \dfrac{1}{2}\cdot\sum_{cyc}\dfrac{1}{\sin^2 A+\sin^2 B} \underset{(2)}{\ge} \dfrac{1}{2}\cdot\dfrac{9}{2\cdot\sum_{cyc}\sin^2 A} \underset{(3)}{\ge} \dfrac{9}{4\cdot\dfrac{9}{4}}=1$,

egalitatea are loc dacă triunghiul este echilateral.

Solution 10 de ***Mihai Miculița- Romania***

Folosind teorema sinusurilor şi apoi inegalitatea: $\boxed{a^2+b^2\ge 2ab}\Leftrightarrow (a-b)^2\ge 0$, obţinem că:

$$\sin A+\sin B=\frac{a}{2R}+\frac{b}{2R}=\frac{a+b}{2R}\Rightarrow(\sin A+\sin B)^2=\frac{(a+b)^2}{4R^2}=$$

$$=\frac{a^2+b^2+2ab}{4R^2}\le\frac{2\left(a^2+b^2\right)}{4R^2}=2.\frac{a^2+b^2}{(2R)^2}\Rightarrow\frac{1}{\sin A+\sin B}\ge\frac{1}{2}\cdot\frac{(2R)^2}{a^2+b^2}. \quad (1)$$

În mod analog, arătăm că: $\dfrac{1}{\sin A+\sin C}\ge\dfrac{1}{2}\cdot\dfrac{(2R)^2}{a^2+c^2}$; (2) şi $\dfrac{1}{\sin B+\sin C}\ge\dfrac{1}{2}\cdot\dfrac{(2R)^2}{b^2+c^2}$. (3)

Adunând acum inegalităţile (1), (2) şi (3), membru cu membru şi folosind apoi inegalitatea:

$$\mathbf{I.}\boxed{\frac{x^2}{m}+\frac{y^2}{n}+\frac{z^2}{p}\ge\frac{(x+y+z)^2}{m+n+p};(\forall)x,y,z,m,n,p\in\mathbb{R};\ m,n,p>0},$$

obţinem că:

$$\frac{1}{(\sin A+\sin B)^2}+\frac{1}{(\sin A+\sin C)^2}+\frac{1}{(\sin B+\sin C)^2}\ge\frac{1}{2}\cdot\left[\frac{(2R)^2}{a^2+b^2}+\frac{(2R)^2}{a^2+c^2}+\frac{(2R)^2}{b^2+c^2}\right]\ge$$

$$\ge\frac{1}{2}.\frac{(6R)^2}{2\left(a^2+b^2+c^2\right)}=\frac{36R^2}{4.\left(a^2+b^2+c^2\right)}=\frac{9R^2}{a^2+b^2+c^2}\ge 1 \text{ (fiindcă: } \mathbf{II.}\boxed{9R^2\ge a^2+b^2+c^2})\Rightarrow$$

$$\Rightarrow\boxed{\frac{1}{(\sin A+\sin B)^2}+\frac{1}{(\sin A+\sin C)^2}+\frac{1}{(\sin B+\sin C)^2}\ge 1}.\blacksquare$$

Demonstraţia inegalităţilor I şi II, folosite în soluţia dată:

I. Avem: $\frac{x^2}{m}+\frac{y^2}{n}\geq\frac{(x+y)^2}{m+n}\Big|\cdot mn(m+n)\Leftrightarrow(m+n)\left(nx^2+my^2\right)\geq mn(x+y)^2\Leftrightarrow$

$$\Leftrightarrow \cancel{mnx^2}+n^2x^2+m^2y^2+\cancel{mny^2}\geq\cancel{mnx^2}+2mnxy+\cancel{mny^2}\Leftrightarrow$$

$$\Leftrightarrow n^2x^2-2mnxy+m^2y^2\geq 0\Leftrightarrow(nx-my)^2\geq 0.$$

Aşa că: $\frac{x^2}{m}+\frac{y^2}{n}\geq\frac{(x+y)^2}{m+n}\Rightarrow\left(\frac{x^2}{m}+\frac{y^2}{n}\right)+\frac{z^2}{p}\geq\frac{(x+y)^2}{m+n}+\frac{z^2}{p}\geq\frac{(x+y+z)^2}{m+n+p}$. ■

II. Notând cu $G-$centrul cercului circumscris şi cu $G-$centrul de greutate al triunghiului ABC, avem: $|OA|=|OB|=|OC|=R$ şi din: $\overrightarrow{OG}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}\Rightarrow 9.|OG|^2=\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)^2=$

$$=|OA|^2+|OB|^2+|OC|^2+2.\left(\overrightarrow{OA}\cdot\overrightarrow{OB}+\overrightarrow{OA}\cdot\overrightarrow{OC}+\overrightarrow{OB}\cdot\overrightarrow{OC}\right);\quad(1)$$

Însă, din: $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}\Rightarrow c^2=|AB|^2=\left(\overrightarrow{OB}-\overrightarrow{OA}\right)^2=|OA|^2-2.\overrightarrow{OB}\cdot\overrightarrow{OA}+|OB|^2=$

$$=2R^2-2.\overrightarrow{OB}\cdot\overrightarrow{OA}\Rightarrow\boxed{2.\overrightarrow{OA}\cdot\overrightarrow{OB}=2R^2-c^2};\quad(2)$$

şi analoagele: $\boxed{2.\overrightarrow{OA}\cdot\overrightarrow{OC}=2R^2-b^2}$; (3) şi $\boxed{2.\overrightarrow{OB}\cdot\overrightarrow{OC}=2R^2-a^2}$. (4)

În fine, ţinând acum seama de relaţiile (2), (3) şi (4), relaţia (1) devine:

$9.|OG|^2=9R^2-\left(a^2+b^2+c^2\right)\geq 0\Leftrightarrow\boxed{9R^2\geq a^2+b^2+c^2}$. ■

Solution 11 de ***Constantin Telteu***, *C.N.A. „Regina Maria”, Constanța - Romania*

$$S=\frac{1}{(\sin A+\sin B)^2}+\frac{1}{(\sin B+\sin C)^2}+\frac{1}{(\sin C+\sin A)^2}=\frac{4R^2}{(a+b)^2}+\frac{4R^2}{(b+c)^2}+\frac{4R^2}{(c+a)^2}$$ - am folosit teorema sinusurilor

Folosind inegalitatea lui Titu Andreescu (Harald Bergström – Nota Redacţiei), obținem:

$$S\geq\frac{36R^2}{(a+b)^2+(b+c)^2+(c+a)^2}=\frac{36R^2}{2\left(a^2+b^2+c^2+ab+bc+ca\right)}$$

Cu inegalitatea (1) $a^2+b^2+c^2\leq 9R^2$ și cu inegalitatea $a^2+b^2+c^2\geq ab+bc+ca$, obținem:

$$S\geq\frac{36R^2}{18R^2+2(ab+bc+ca)}\geq\frac{36R^2}{18R^2+2\left(a^2+b^2+c^2\right)}\geq\frac{36R^2}{36R^2}=1\quad\text{c.c.t.d.}$$

La toate inegalitățile folosite, egalitatea se obține pentru triunghiul echilateral, deci și pentru inegalitatea din enunț.

Pentru demonstrarea inegalității (1) , folosim mai întâi egalitatea

(2) $\quad a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right)$ cu care inegalitatea (1) devine:

$$2\left(p^2-r^2-4Rr\right)\leq 9R^2\Leftrightarrow 2p^2\leq 9R^2+2r^2+8Rr$$

Pentru demonstrarea ultimei inegalități folosim inegalitatea lui Gerretsen $p^2 \leq 4R^2 + 4Rr + 3r^2$ și inegalitatea lui Euler $R \geq 2r$, sub forma $R^2 \leq 4r^2$. Avem:

$2p^2 \leq 2\left(4R^2 + 4Rr + 3r^2\right) \leq 8R^2 + 8Rr + 6r^2 \leq 8R^2 + 8Rr + R^2 + 2r^2 = 9R^2 + 8Rr + 2r^2$.

Pentru demonstrarea egalității (2), putem proceda astfel:

Luăm $a = x + y,\ b = y + z,\ c = z + x,\ \ x, y, z \geq 0$ și cu acestea avem:

$p = x + y + z,\ r = \sqrt{\dfrac{xyz}{x + y + z}},\ R = \dfrac{(x+y)(y+z)(z+x)}{4\sqrt{(x+y+z)xyz}}$. Înlocuindu-le pe acestea în (2) se verifică egalitatea.

Solution 12 de **Rotariu Gheorghe – Romania**

Din teorema sinusurilor avem, cu notaţiile uzuale,

$$\sin A = \frac{a}{2R};\quad \sin B = \frac{b}{2R};\quad \sin C = \frac{c}{2R}.$$

Inegalitatea de demonstrat devine

$$\left(\frac{2R}{a+b}\right)^2 + \left(\frac{2R}{b+c}\right)^2 + \left(\frac{2R}{c+a}\right)^2 \geq 1 \Leftrightarrow 4R^2\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq 1.$$

Din inegalitatea Titu-Andreescu (Harald Bergström – Nota Redacţiei), avem

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{(1+1+1)^2}{(a+b)^2 + (b+c)^2 + (c+a)^2} =$$

$$= \frac{9}{2(a^2 + b^2 + c^2 + ab + ac + bc)}. \qquad (1)$$

Deoarece $a^2 + b^2 + c^2 \geq ab + ac + bc$, avem

$$a^2 + b^2 + c^2 + ab + ac + bc \leq 2(a^2 + b^2 + c^2). \qquad (2)$$

Cu acestea, avem

$$4R^2\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \overset{(1)}{\geq} \frac{4R^2 \cdot 9}{2(a^2 + b^2 + c^2 + ab + ac + bc)} \geq$$

$$\overset{(2)}{\geq} \frac{4R^2 \cdot 9}{4(a^2 + b^2 + c^2)} = \frac{9R^2}{a^2 + b^2 + c^2} \geq 1,$$

deoarece $9R^2 \geq a^2 + b^2 + c^2$ (avem cunoscuta egalitate $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$).

Solution 13 *Marin Chirciu - Colegiul Național „Zinca Golescu”, Pitești - Romania*

Folosind teorema sinusurilor avem

$$\sum_{cyc}\frac{1}{(\sin B+\sin C)^2}=4R^2\sum_{cyc}\frac{1}{(b+c)^2}\text{, iar }\sum_{cyc}\frac{1}{(b+c)^2}=\frac{9p^4+2p^2(4Rr-3r^2)+r^2(4R+r)^2}{4p^2(p^2+r^2+2Rr)^2},$$

care rezultă din

$$\sum_{cyc}\frac{1}{(b+c)^2}=\frac{\sum_{cyc}(a+b)^2(a+c)^2}{\prod_{cyc}(b+c)^2}=\frac{\sum_{cyc}(a^2+ab+bc+ca)^2}{\left[2p(p^2+r^2+2Rr)\right]^2}=\frac{\sum_{cyc}(a^2+p^2+r^2+4Rr)^2}{\left[2p(p^2+r^2+2Rr)\right]^2},$$

apoi cu identitățile cunoscute în triunghi $\sum_{cyc}a^4=2\left[p^4-2p^2(4Rr+3r^2)+r^2(4R+r)^2\right]$ și

$\sum_{cyc}a^2=2(p^2-r^2-4Rr)$ deducem formula sumei $\sum_{cyc}\frac{1}{(b+c)^2}$.

Obținem formula $\sum_{cyc}\frac{1}{(\sin B+\sin C)^2}=\frac{R^2}{p^2}\cdot\frac{9p^4+2p^2(4Rr-3r^2)+r^2(4R+r)^2}{(p^2+r^2+2Rr)^2}$.

Inegalitatea de demonstrat se scrie $\frac{R^2}{p^2}\cdot\frac{9p^4+2p^2(4Rr-3r^2)+r^2(4R+r)^2}{(p^2+r^2+2Rr)^2}\ge 1$.

Ținând seama de inegalitatea lui Mitrinović $p\le\frac{3R\sqrt{3}}{2}$, rezultă $\frac{R^2}{p^2}\ge\frac{4}{27}$.

Este suficient să arătăm că $\frac{4}{27}\cdot\frac{9p^4+2p^2(4Rr-3r^2)+r^2(4R+r)^2}{(p^2+r^2+2Rr)^2}\ge 1\Leftrightarrow$

$$4\left[9p^4+2p^2(4Rr-3r^2)+r^2(4R+r)^2\right]\ge 27(p^2+r^2+2Rr)^2\Leftrightarrow$$

$$9p^4-p^2(76Rr+78r^2)\ge r^2(44R^2+76Rr+23r^2)\Leftrightarrow$$

$$p^2(9p^2-76Rr-78r^2)\ge r^2(44R^2+76Rr+23r^2).$$

Cum $(9p^2-76Rr-78r^2)>0$, din inegalitatea lui Gerresten $p^2\ge 16Rr-5r^2$ și inegalitatea lui Euler $R\ge 2r$, este suficient să arătăm că

$$(16Rr-5r^2)\left[9(16Rr-5r^2)-76Rr-78r^2\right]\ge r^2(44R^2+76Rr+23r^2)\Leftrightarrow$$

$$(16R-5r)(68R-23r)\ge 44R^2+76Rr+23r^2\Leftrightarrow\ 1044R^2-2384Rr+592r^2\ge 0\ \Leftrightarrow$$

$261R^2 - 596Rr + 148r^2 \geq 0 \Leftrightarrow (R-2r)(261R-74r) \geq 0$, evident din inegalitatea lui Euler.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Notă.

Inegalitatea poate fi întărită:

Prove that in all triangles ABC holds the inequality

$$\sum_{cyc} \frac{1}{(\sin B + \sin C)^2} \geq \frac{27R^2}{4p^2} .$$

Marin Chirciu, Pitești

Soluție

Folosind formula demonstrată $\sum_{cyc} \frac{1}{(\sin B + \sin C)^2} = \frac{R^2}{p^2} \cdot \frac{9p^4 + 2p^2(4Rr - 3r^2) + r^2(4R+r)^2}{(p^2 + r^2 + 2Rr)^2}$,

inegalitatea de demonstrat se scrie $\frac{R^2}{p^2} \cdot \frac{9p^4 + 2p^2(4Rr - 3r^2) + r^2(4R+r)^2}{(p^2 + r^2 + 2Rr)^2} \geq \frac{27R^2}{4p^2} \Leftrightarrow$

$4\left[9p^4 + 2p^2(4Rr - 3r^2) + r^2(4R+r)^2\right] \geq 27(p^2 + r^2 + 2Rr)^2 \Leftrightarrow$

$9p^4 - p^2(76Rr + 78r^2) \geq r^2(44R^2 + 76Rr + 23r^2) \Leftrightarrow$

$p^2(9p^2 - 76Rr - 78r^2) \geq r^2(44R^2 + 76Rr + 23r^2)$.

Cum $(9p^2 - 76Rr - 78r^2) > 0$, din inegalitatea lui Gerresten $p^2 \geq 16Rr - 5r^2$ și inegalitatea lui Euler $R \geq 2r$, este suficient să arătăm

că $(16Rr - 5r^2)\left[9(16Rr - 5r^2) - 76Rr - 78r^2\right] \geq r^2(44R^2 + 76Rr + 23r^2) \Leftrightarrow$

$(16R - 5r)(68R - 23r) \geq 44R^2 + 76Rr + 23r^2 \Leftrightarrow 1044R^2 - 2384Rr + 592r^2 \geq 0 \Leftrightarrow$

$261R^2 - 596Rr + 148r^2 \geq 0 \Leftrightarrow (R-2r)(261R-74r) \geq 0$, evident din inegalitatea lui Euler.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Observație.

Ținând seama de inegalitatea lui Mitrinović $p \leq \frac{3R\sqrt{3}}{2}$, rezultă $\frac{27R^2}{4p^2} \geq 1$,de unde

$\sum_{cyc} \frac{1}{(\sin B + \sin C)^2} \geq \frac{27R^2}{4p^2} \geq 1$, adică inegalitatea $\sum_{cyc} \frac{1}{(\sin B + \sin C)^2} \geq \frac{27R^2}{4p^2}$ este mai tare decât inegalitatea $\sum_{cyc} \frac{1}{(\sin B + \sin C)^2} \geq 1$.

Solution 14 de **Nela Ciceu şi Roxana Mihaela Stanciu**

Prezența expresiilor de tipul $(\sin A + \sin B)^2$ la numitorii fracțiilor ne sugerează o inegalitate foarte cunoscută - Iran 1996:

Dacă a, b, c > 0, atunci $$(ab+bc+ca)\sum\frac{1}{(a+b)^2}\geqslant\frac{9}{4} \qquad (*)$$

Deoarece vom folosi această inegalitate în cazul particular când a, b, c sunt laturile unui triunghi, iată o soluție elementară a inegalității (*) în acest caz (preluată din [2]):

$$\sum\frac{ab+bc+ca}{(a+b)^2}-\frac{9}{4}=\sum\left(\frac{ab+bc+ca}{(a+b)^2}-\frac{3}{4}\right)=\sum\frac{(a+3b)(c-b)+(3a+b)(c-a)}{4(a+b)^2}=$$
$$=\sum\frac{(a+3b)(c-b)}{4(a+b)^2}+\sum\frac{(3a+b)(c-a)}{4(a+b)^2}=\sum\frac{(a+3b)(c-b)}{4(a+b)^2}+\sum\frac{(3c+a)(b-c)}{4(c+a)^2}=$$
$$=\sum\frac{(b-c)[(a+b)^2(3c+a)-(c+a)^2(a+3b)]}{4(a+b)^2(c+a)^2}=\sum\frac{(b-c)^2[3bc+a(b+c-a)]}{4(a+b)^2(c+a)^2}\geqslant 0.$$

Aplicând inegalitatea (*) pentru sinA, sinB, sinC (care sunt laturile unui triunghi) obținem
$$(\sin A\sin B+\sin B\sin C+\sin C\sin A)\sum_{cyc}\frac{1}{(\sin A+\sin B)^2}\geqslant\frac{9}{4}.$$

Rezultă imediat că este suficient să demonstrăm că

$\sin A\sin B + \sin B\sin C + \sin C\sin A <= 9/4$.

Avem $\sin A\sin B + \sin B\sin C + \sin C\sin A <= \sin^2 A + \sin^2 B + \sin^2 C <= 9/4$, unde ultima inegalitate este itemul 2.3 din [1].

[1] O. Bottema, *Geometric Inequalities*, Groningen 1969
[2] T. Zvonaru, *Inegalități ligamentate și neligamentate*, Arhimede 5-6/2003

Solution 15 de **Buzea Gabriela**, Școala Gimnazială Nr. 56, București

Utilizăm inegalitatea lui Radon, ea fiind un caz particular a inegalității lui Hölder
$$\frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}+\cdots+\frac{x_n^{p+1}}{a_n^p}\geq\frac{(x_1+x_2+\cdots+x_n)^{p+1}}{(a_1+a_2+\cdots+a_n)^p}$$
unde $x_1,x_2,...,x_n$ și $a_1,a_2,...,a_n$ sunt numere reale pozitive si p numar real pozitiv.

Pentru cazul particular n=3 și p=2 obținem inegalitatea

$$\frac{x_1^3}{a_1^2}+\frac{x_2^3}{a_2^2}+\frac{x_3^3}{a_3^2}\geq\frac{(x_1+x_2+x_3)^3}{(a_1+a_2+a_3)^2} \quad (1)$$

Aplicând inegalitatea (1) rezultă

$$\frac{1}{(sinA+sinB)^2}+\frac{1}{(\sin B+sinC)^2}+\frac{1}{(sinC+sinA)^2}\geq$$
$$\geq\frac{(1+1+1)^3}{(2sinA+2sinB+2sinC)^2}=\frac{27}{4(sinA+sinB+sinC)^2}$$

Din teorema sinusurilor avem

$$sinA=\frac{a}{2R}, \qquad sinB=\frac{b}{2R}, \qquad sinC=\frac{c}{2R}.$$

Atunci

$$sinA+sinB+sinC=\frac{a+b+c}{2R}\leq\frac{3\sqrt{3}R}{2R}=\frac{3\sqrt{3}}{2} \quad (2)$$

deoarece $(a+b+c)^2\leq 3(a^2+b^2+c^2), conform\ inegalității\ C-B-S,$ iar $a^2+b^2+c^2\leq 9R^2$, conform inegalității lui Leibniz.

Folosind inegalitatea (2) obținem că

$$\frac{27}{4(sinA+sinB+sinC)^2}\geq\frac{27}{4\cdot\left(\frac{3\sqrt{3}}{2}\right)^2}=1$$

Deci,

$$\sum\frac{1}{(sinA+sinB)^2}\geq 1$$

evident cu egalitate dacă triunghiul ABC este echilateral.

Mulțumim profesorilor rezolvitori pentru soluțiile interesante și tuturor celor care susțin Revista Electronică MateInfo.ro !

Mulțumim, de asemenea, domnului profesor Dan Sitaru pentru susținerea acestui concurs și pentru soluțiile trimise !

Cu stimă, Andrei Octavian Dobre

3. STUDIU DE SPECIALITATE ASUPRA VARIAŢIEI DISTANŢEI DINTRE PUNCTELE UNUI PLAN ŞI UN PUNCT, O DREAPTĂ SAU UN PLAN

Prof. Stan Ilie
Colegiul Tehnic "Anghel Saligny", Roşiorii de Vede, Teleorman

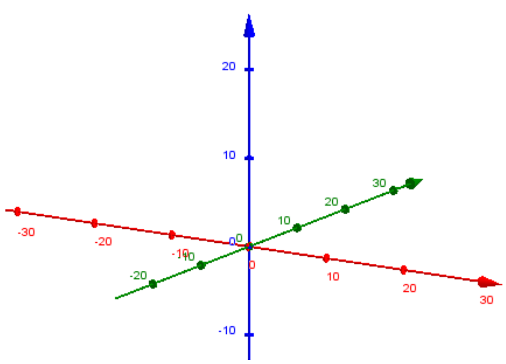
Actualul studiu este o continuare a articolului din numărul precedent „Studiu de specialitate asupra variaţiei distanţei dintre punctele unei drepte şi un punct, o dreaptă sau un plan”:
http://mateinfo.ro/reviste-de-matematica/revista-electronica-de-matematica-mateinfo-ro-aparitie-lunara-2065-6432/revista-mateinfo-ro/revista-electronica-mateinfo-ro-2017/693-revista-electronica-mateinfo-ro-martie-2017

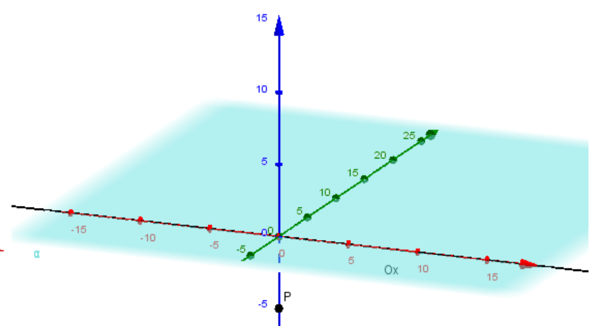
Ne propunem analiza variaţiei distanţei dintre punctele unui plan(**...α**) şi un element geometric elementar(**P**unct, **d**reaptă, pl**α**n) în toate poziţiile lor relative.

De la punctele unui plan **α** la un punct P (...**α**P)

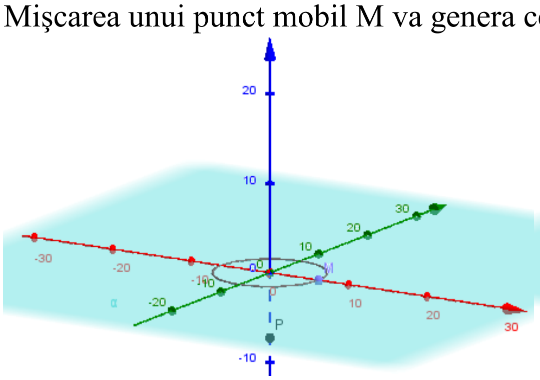
Punct exterior planului

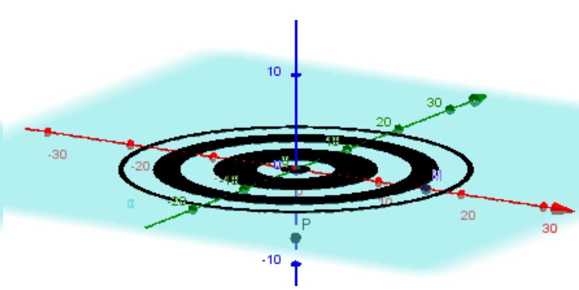
În sistemul cartezian 3D alegem planul **α**=(Ox, Oy) şi **P** pe axa Oz.



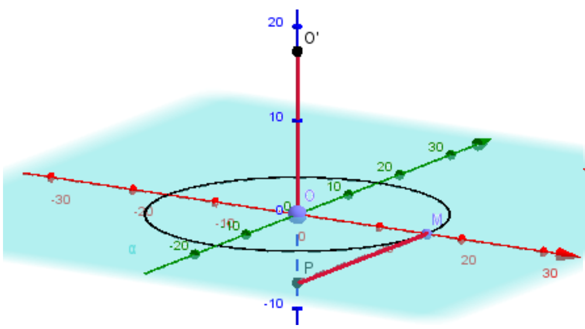


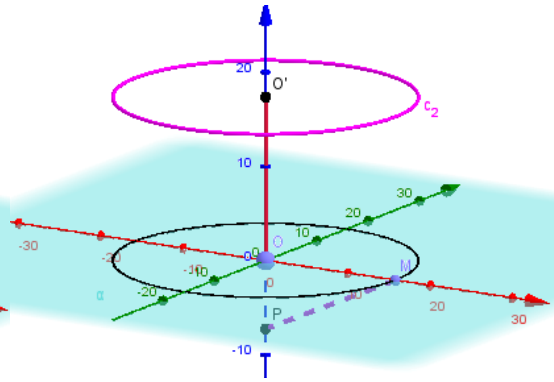
Fie M pe axa Ox şi cercul C_1 de centru O şi rază OM.
Mişcarea unui punct mobil M va genera cercurile $C_1(O, OM)$ care vor acoperi întreg planul **α**.



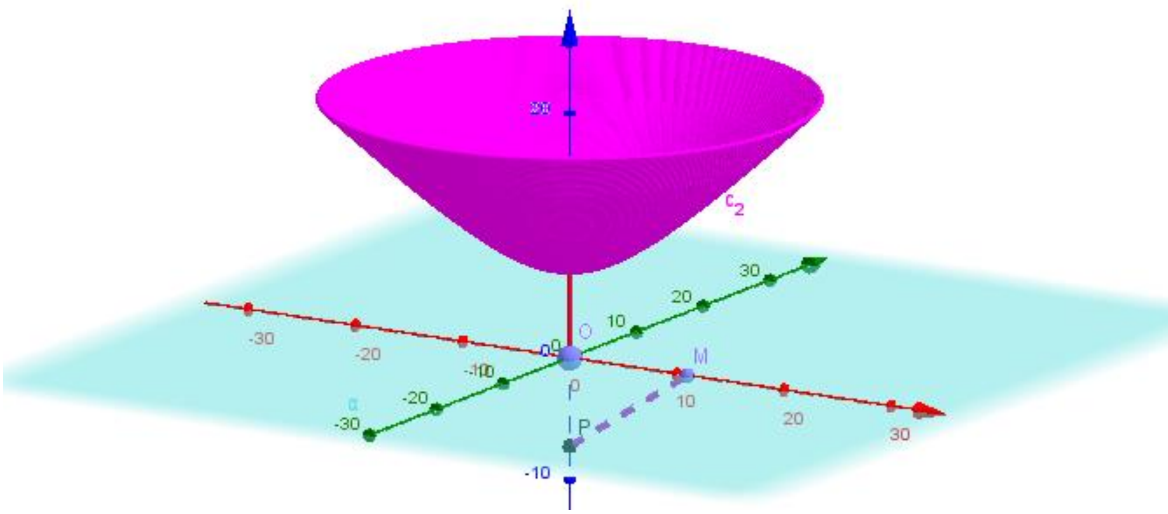


Fie O’ pe Oz la distanţa O’O=PM. Construim, într-un plan paralel cu **α,** cercul C_2 de rază OM. Proiecţia lui C_2 pe **α** este C_1**.** Cotele punctelor lui C_2 reprezintă distanţele de la punctele lui C_1 la P.



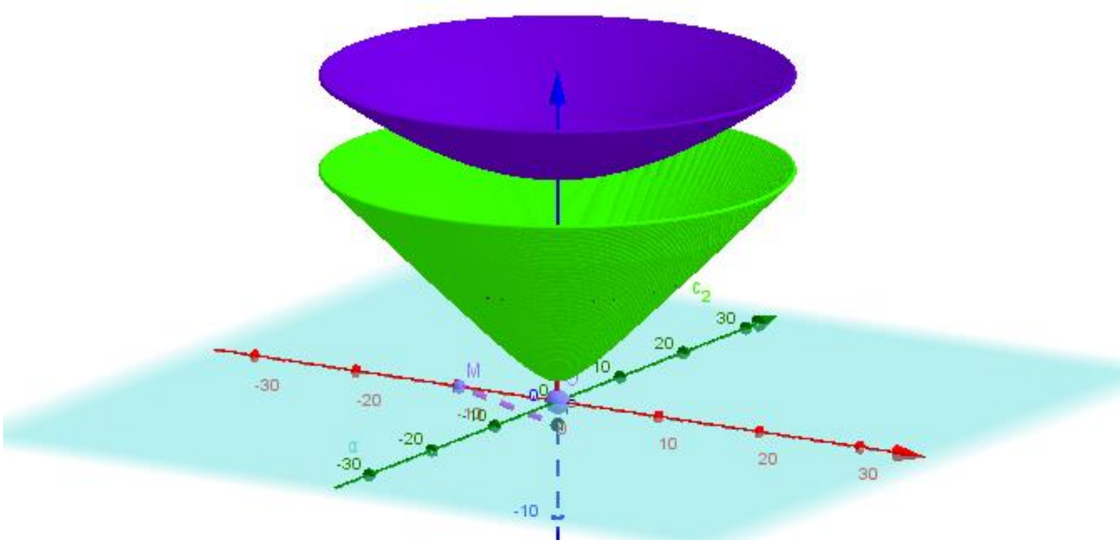


Prin variaţia punctului M pe Ox obţinem…o suprafaţă de rotaţie…



…un hiperboloid cu două pânze! Doar una este reprezentată şi reprezentativă, în acest caz. Rezultatul este relativ uşor demonstrabil matematic.
Ilustrarea poate fi obţinută şi prin alte construcţii ce asigură parcurgerea planului **α**(drepte paralele cu Ox sau dreptele ce trec prin origine).

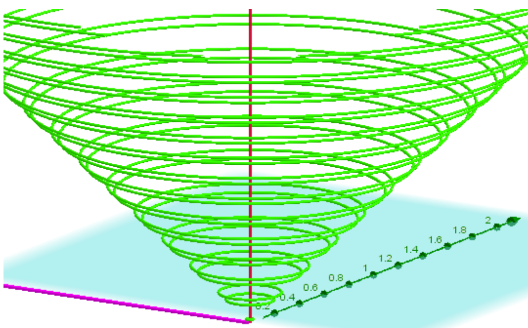
Prin mişcarea lui P în diverse poziţii pe Oz, se obţin diverşi hiperboloizi de rotaţie.

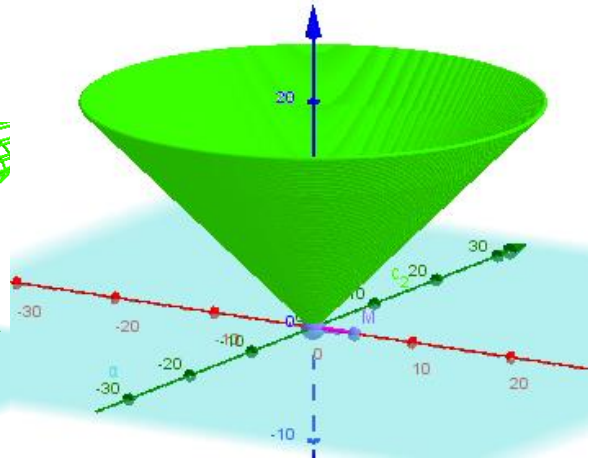


c2
30
20
10
-30
-20
-10
0
10
20
30
-20
-30
-10
M

Punct interior planului

Când punctul P este pe planul **α**(P=O), hiperboloidul degenerează în con!

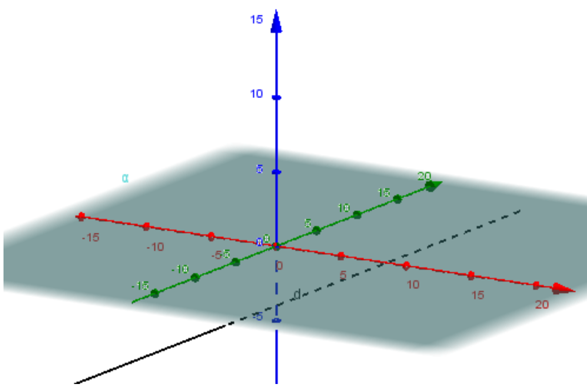


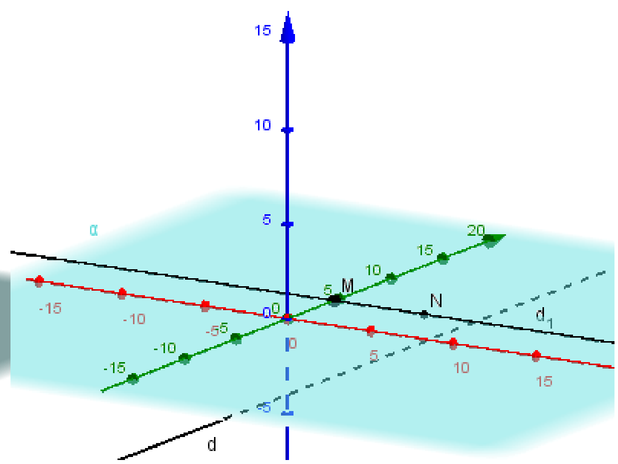


De la punctele unui plan la o dreaptă (...**α**d)

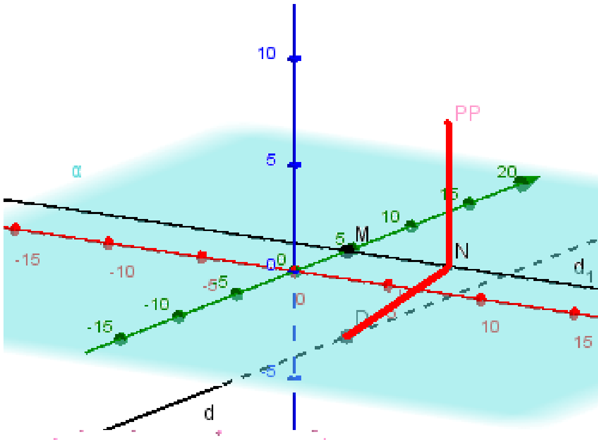
Dreapta şi planul paralele

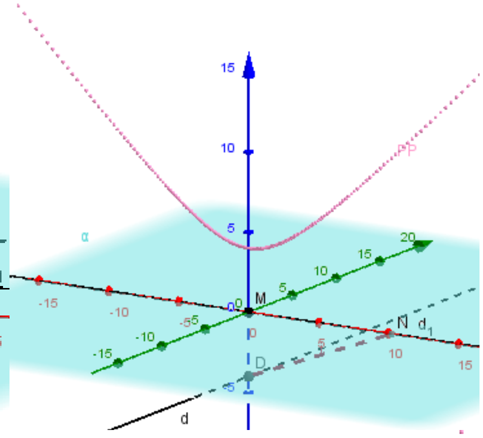
În sistemul cartezian 3D folosim planul xOy ca fiind planul **α** şi o dreaptă **d** paralelă cu el. Fie M punct mobil pe Oy şi N punct mobil pe d_1(d_1 paralela la Ox ce trece prin M). N va acoperi întregul plan **α**.

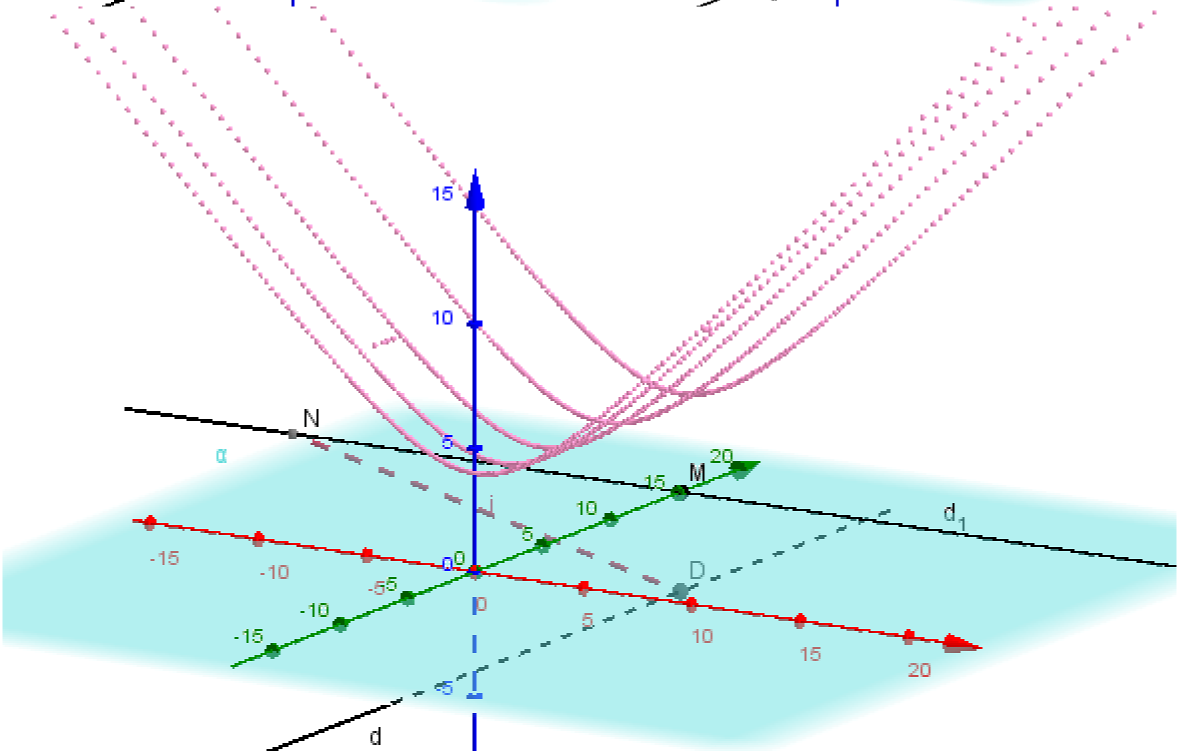


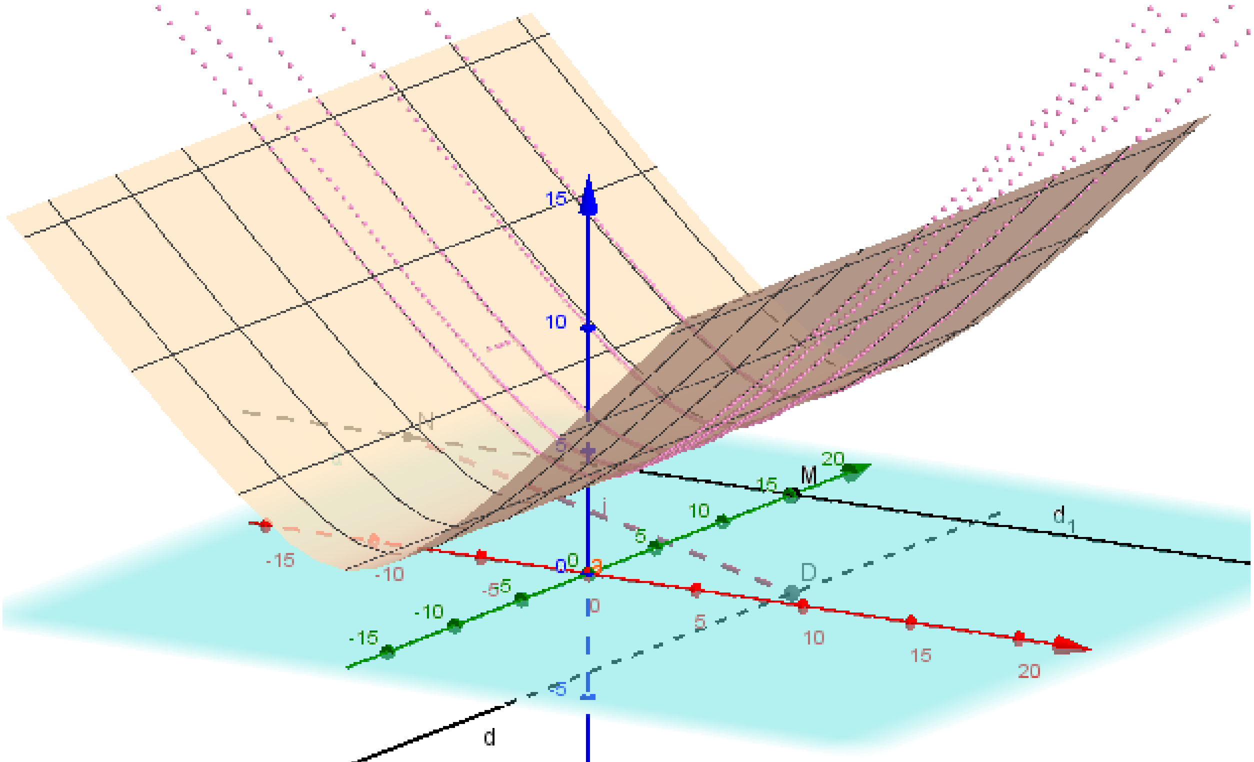


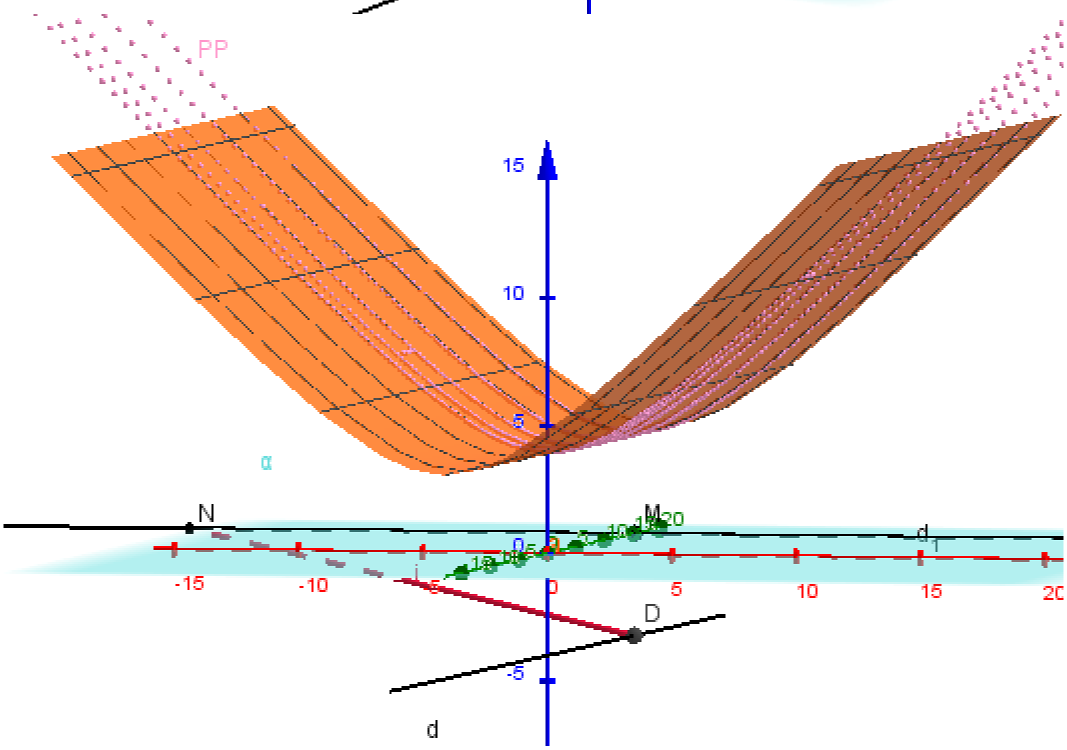
Fie ND perpendicular pe d. Mişcarea lui N pe d_1 (cu M fixat) trasează prin intermediul lui PP o hiperbolă.


PP
α
M
N
D
d
d_1

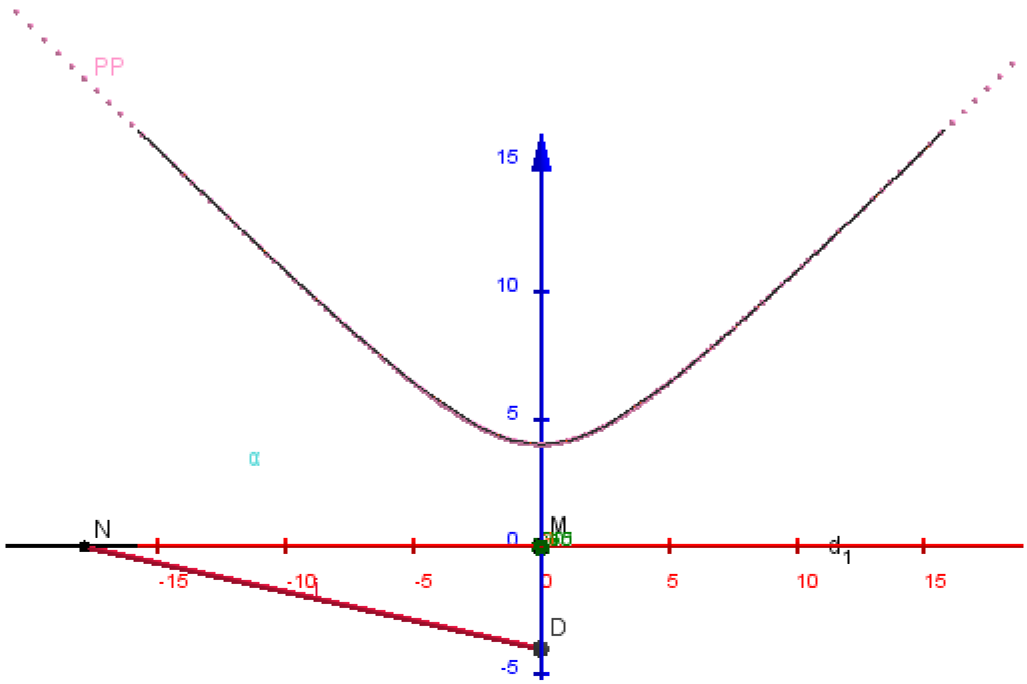

PP
M
N
D
d
d_1


α
N
M
D
d
d_1

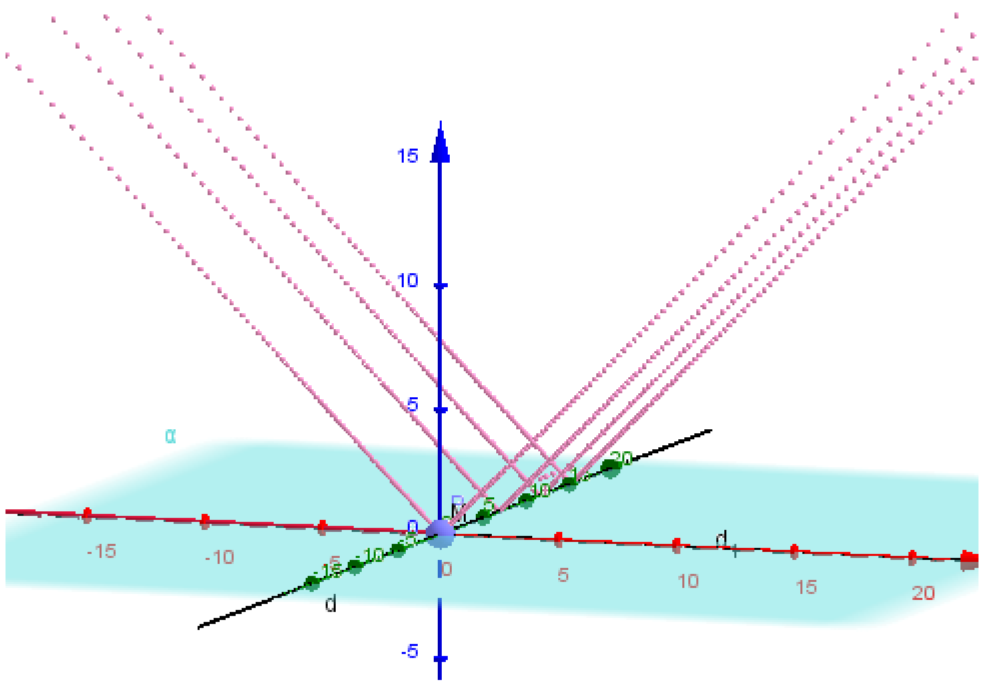


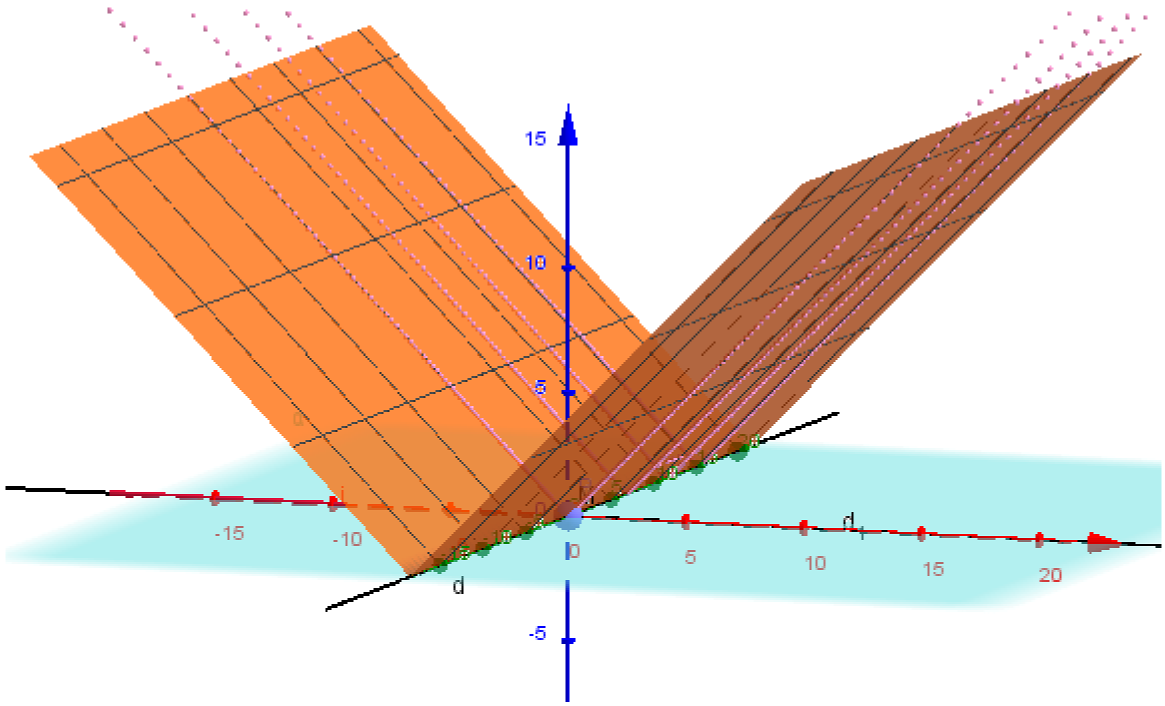


Iar întreaga suprafaţă este o suprafaţă de translaţie...un cilindru hiperbolic.
În secţiune apare astfel:



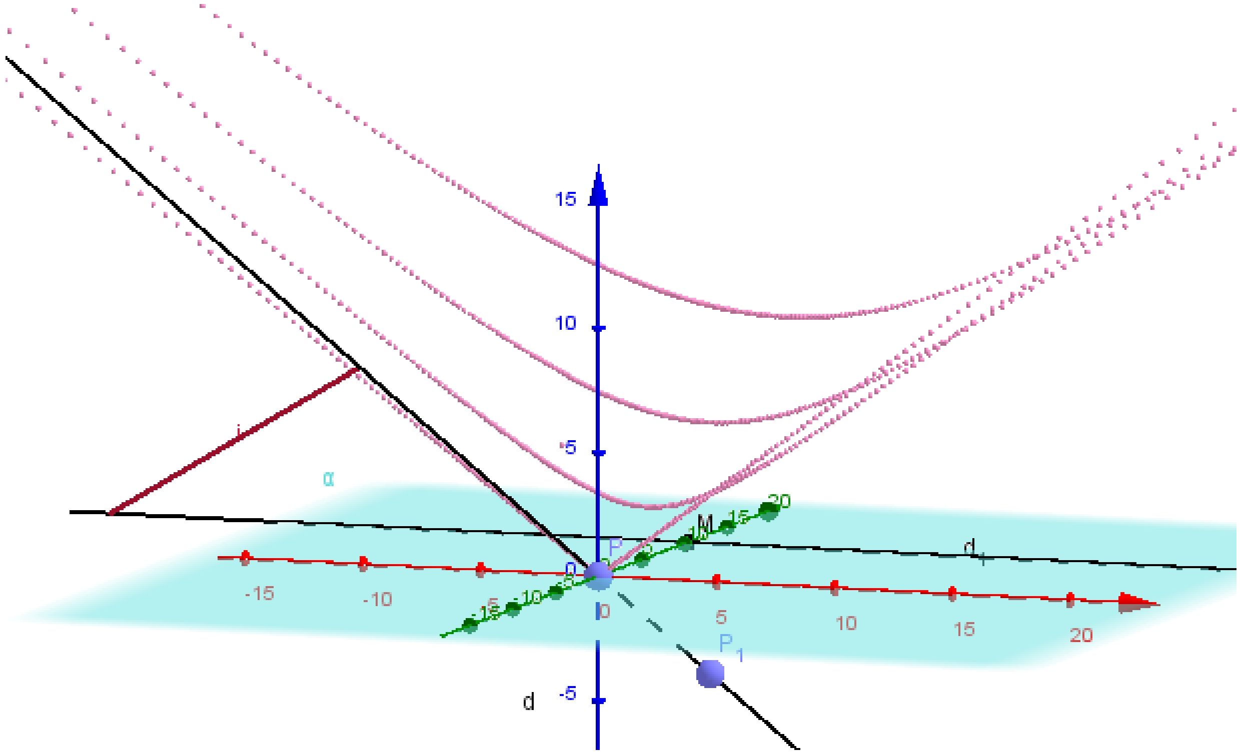
Dreapta conţinută în plan





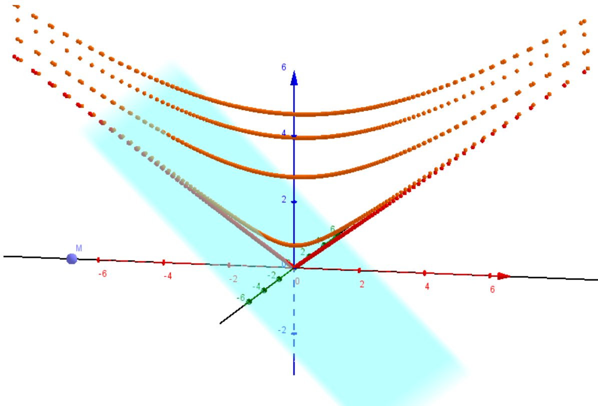
Cilindrul hiperbolic a degenerat în două semiplane(un diedru drept).

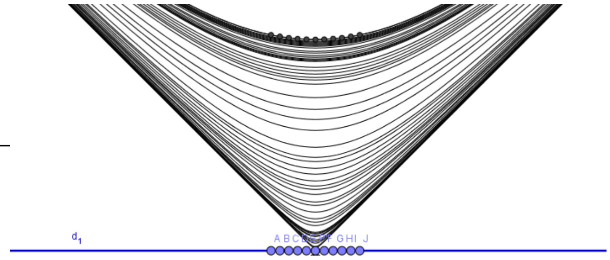
Dreapta şi planul concurente



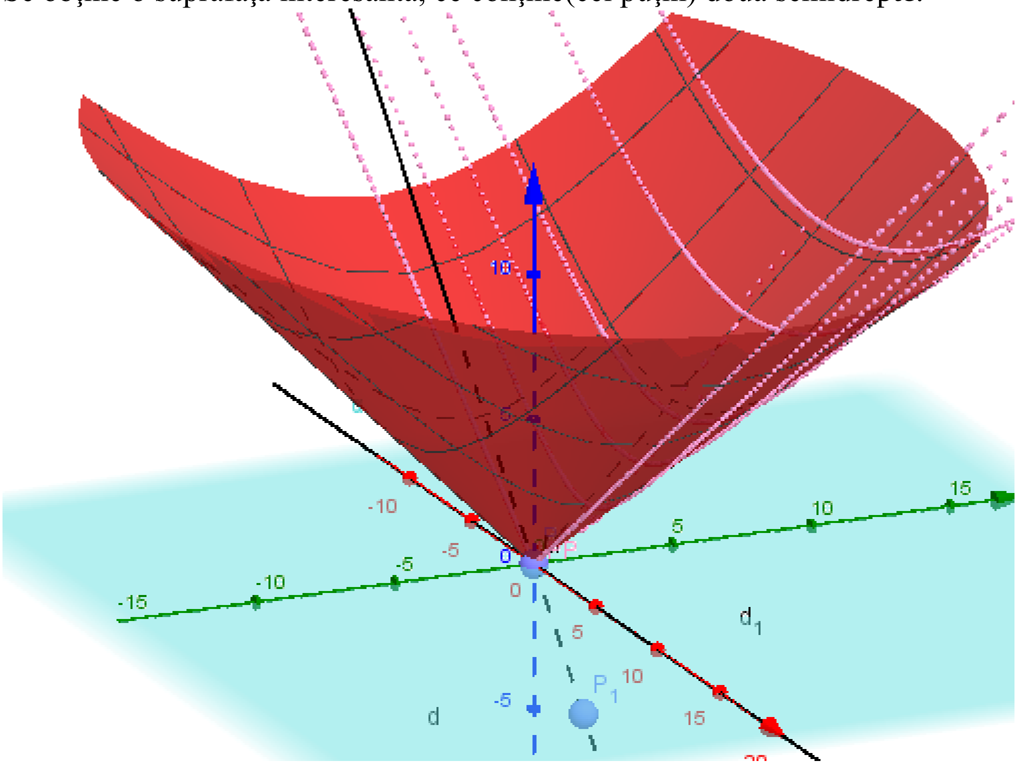
Ce o fi asta?

Parcă ar fi situaţia de la **…dα**, în care hiperbolele sunt "scoase din 2D şi înşirate în 3D"…ca burduful unui acordeon…

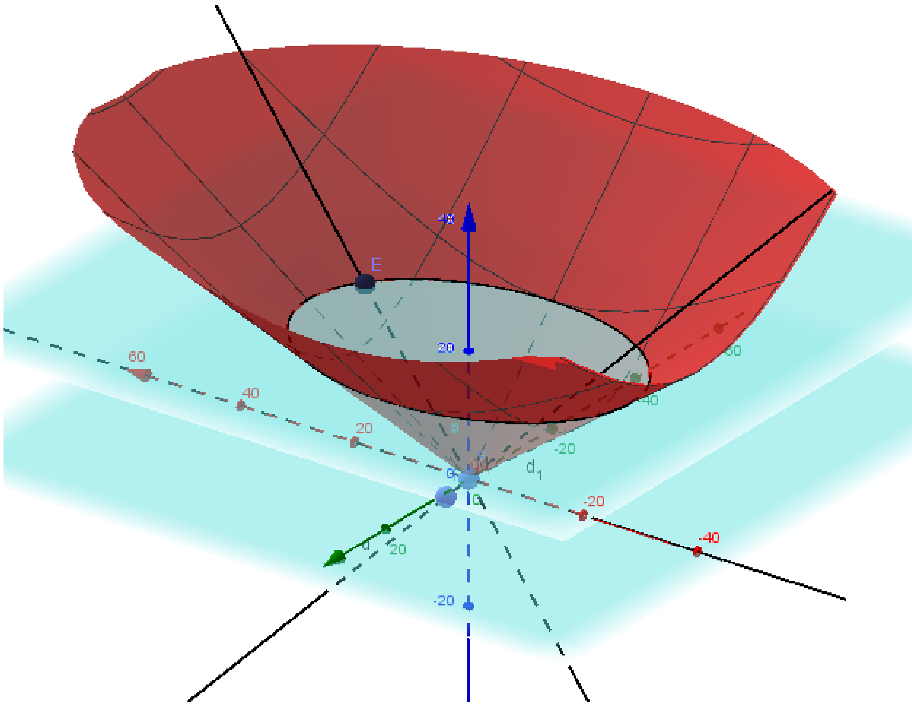




Se obţine o suprafaţă interesantă, ce conţine(cel puţin) două semidrepte.

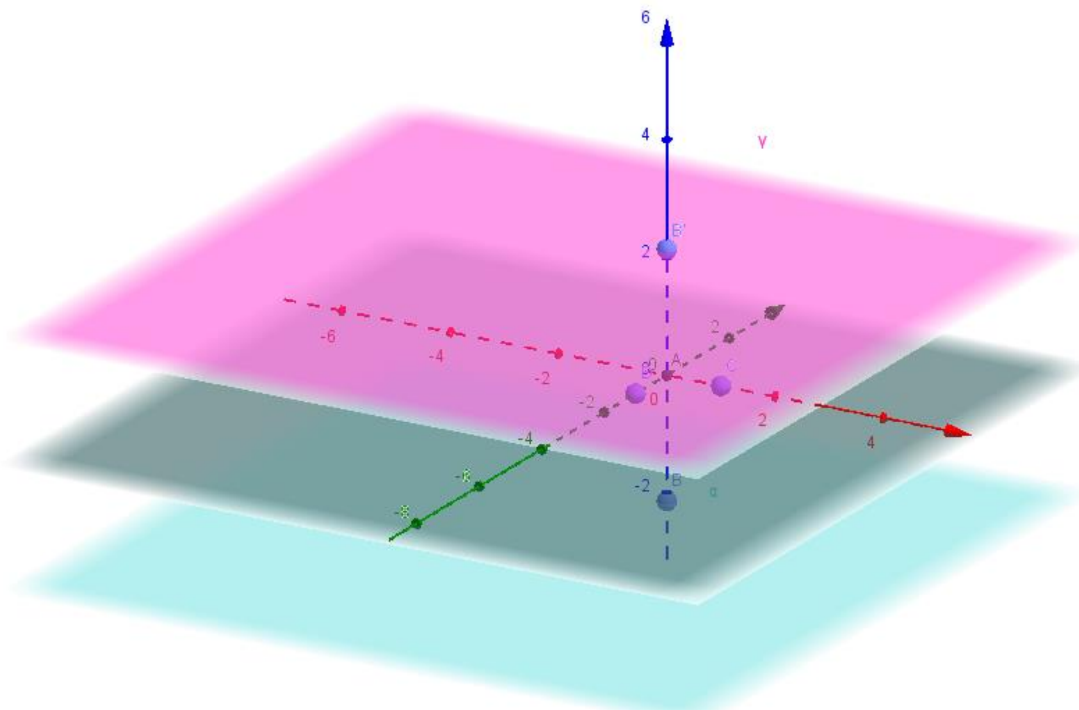


De fapt e o suprafaţă riglată! O suprafaţă conică eliptică! Excentricitatea elipsei este legată de înclinarea dreptei **d** faţă de planul **α.** Când dreapta e perpendiculară pe plan, regăsim cazul **…αP** şi conul circular drept!



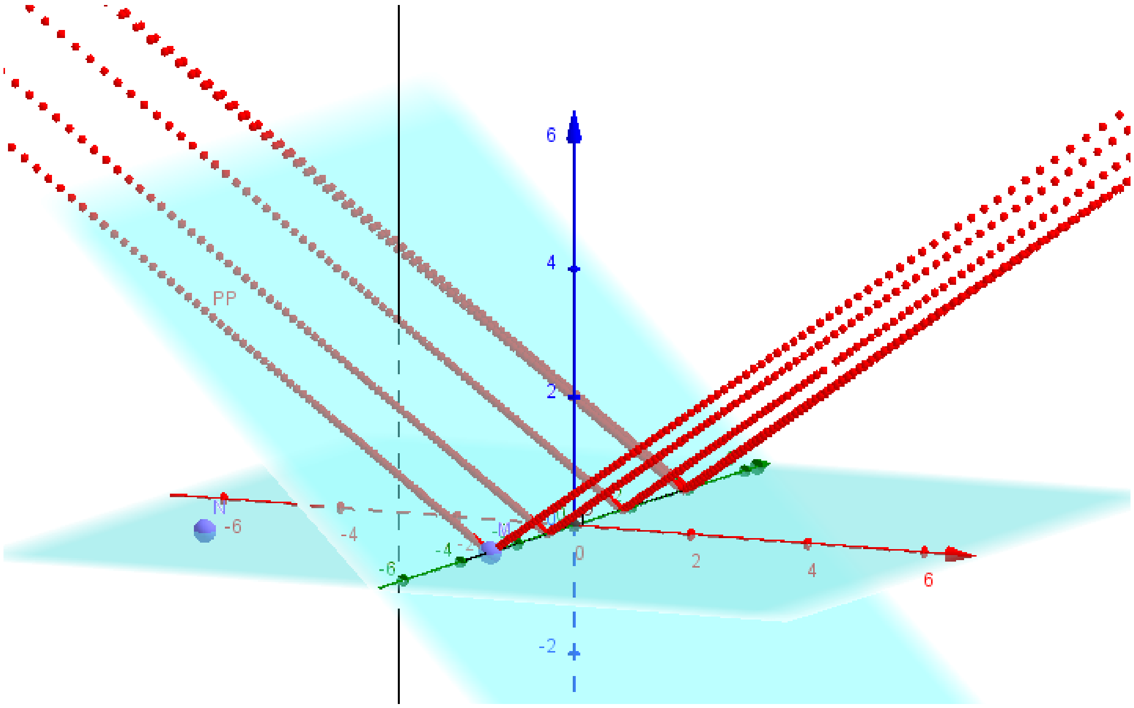
De la punctele unui plan la un plan (...**αα**)

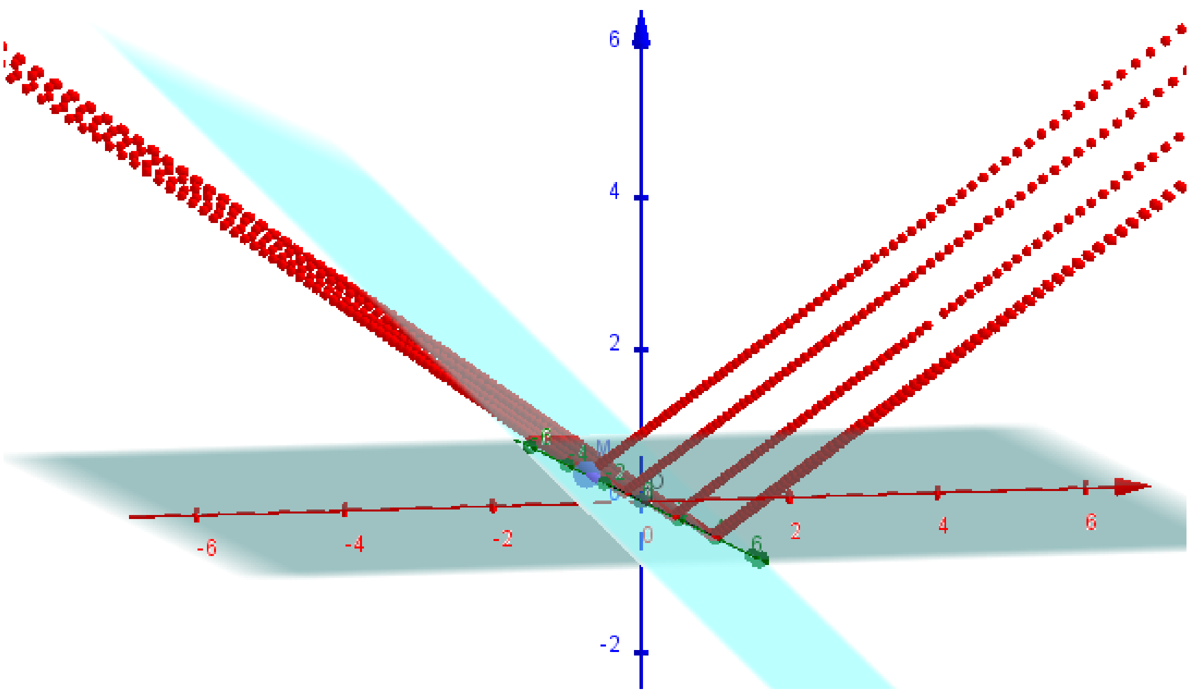
Plane paralele



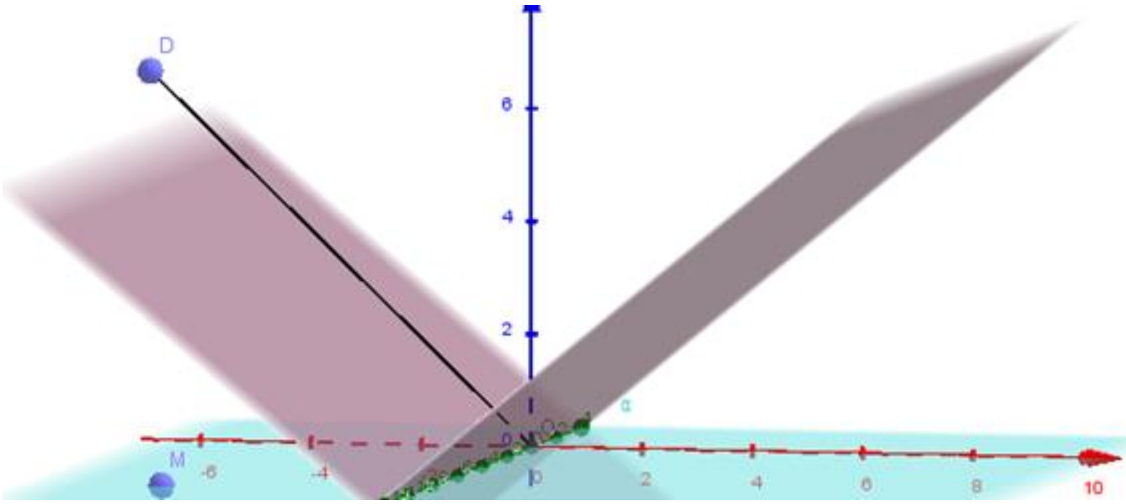
Obţinem simetricul planului α_2 faţă de planul α_1.

Plane concurente

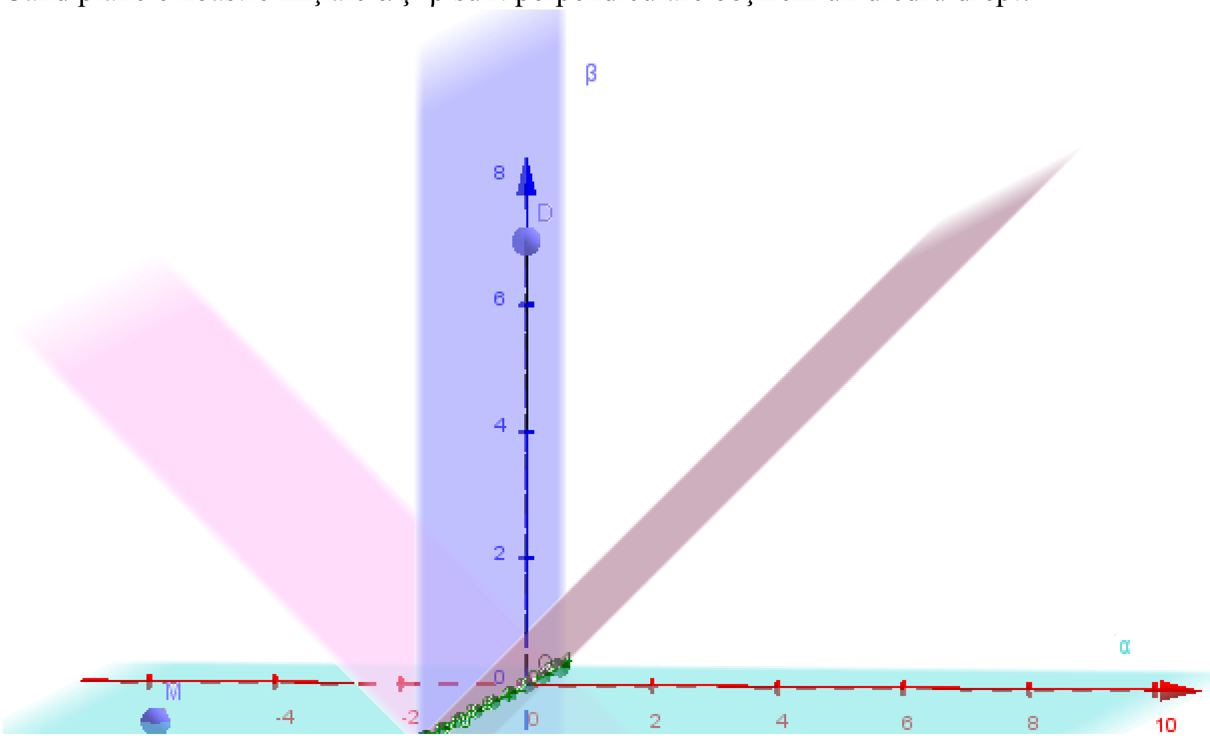




Se obţine un diedru…



Când planele noastre iniţiale α şi β sunt perpendiculare obţinem un diedru drept!



(va urma)

4. ASUPRA PROBLEMEI 4106 DIN CRUX MATHEMATICORUM

Marius Drăgan, Bucureşti

Dacă $m, n \geq 1$ atunci în orice triunghi este adevărată inegalitatea

$$\frac{b^n + c^n}{a^m} + \frac{c^n + a^n}{b^m} + \frac{a^n + b^n}{c^m} \geq 6\left(\frac{3}{2s}\right)^{m-n}.$$

Demonstraţie. $S_n = a^n + b^n + c^n, b^n + c^n = S_n - a^n$, deci $b^n + c^n$ şi $\frac{1}{a^m}$ sunt la fel ordonate.

Cu inegalitatea lui *Cebâşev* şi inegalitatea lui *Jensen* obţinem

$$\sum \frac{b^n + c^n}{a^m} \geq \frac{1}{3}\sum (S_n - a^n)\sum \frac{1}{a^m} = \frac{2}{3} S_n \sum \frac{1}{a^m} =$$

$$= \frac{2}{3}(a^n + b^n + c^n)\sum\left(\frac{1}{a}\right)^m \geq \frac{2}{3}\cdot 3^{1-n}(2s)^n 3^{1-m}\left(\sum \frac{1}{a}\right)^m \geq \frac{6}{3^{m+n}}(2s)^n\left(\frac{9}{2s}\right)^m =$$

$$= \frac{6}{3^{m+n}} \cdot \frac{2^n s^n 3^{2m}}{2^m s^m} = 6\cdot\left(\frac{3}{2}\right)^{m-n}\frac{1}{s^{m-n}}, \ (*).$$

Observaţie.

- Din inegalitatea lui *Mitrinovic*, $s \leq \frac{3\sqrt{3}}{2}R$, rezultă $\frac{1}{s^{m-n}} \geq \frac{2^{m-n}}{3^{\frac{3(m-n)}{2}}} \cdot \frac{1}{R^{m-n}}$, apoi din (*)

  deducem

$$\sum \frac{b^n + c^n}{a^m} \geq \frac{6}{3^{\frac{m-n}{2}}} \cdot \frac{1}{R^{m-n}}.$$

- Pentru $n = 1, m = 5$ se obţine inegalitatea din problema 4106, Crux Mathematicorum, Vol. 42(1), January, 2016.

5. ÎN LEGĂTURĂ CU PROBLEMA 3096 DIN CRUX MATHEMATICORUM, VOL.31(2005),NR.8(OCT)

Marin Chirciu, Colegiul Național „Zinca Golescu”, Pitești

În revista de matematică Crux Mathematicorum din octombrie 2005 este propusă următoarea inegalitate în triunghi:

3096.Let ABC **be a triangle with sides** a,b,c **opposite the angles** A,B,C **, respectively. Prove that**

$$\sum_{cyclic}\frac{bc}{b+c}\sin^2\frac{A}{2}\le\frac{a+b+c}{8}.$$

Proposed by Arkady Alt, San Jose, CA, USA

Articolul prezintă inegalități cu sume de forma $\sum_{cyclic}\frac{bc}{b+c}f(A)$ și $\sum_{cyclic}\frac{bc}{b+c}f(B)f(C)$, unde f este una dintre funcțiile trigonometrice.

1. a) $\sum\frac{bc}{b+c}\sin^2\frac{A}{2}\le\frac{a+b+c}{8}$**.**

Crux Mathematicorum 31/2005, Arkady Alt, USA

Soluţie:

Vom nota cu M_s și M_d membrul stâng și respectiv membrul drept al relației de demonstrat.

Folosim identităţile cunoscute $\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}$ şi $\sum a\sin^2\frac{A}{2}=p\left(1-\frac{r}{R}\right)$.

$$M_s=\sum\frac{bc}{b+c}\sin^2\frac{A}{2}\le\sum\frac{b+c}{4}\sin^2\frac{A}{2}=\frac{1}{4}\sum(b+c)\sin^2\frac{A}{2}=\frac{1}{4}\sum(\underbrace{a+b+c}_{=2p}-a)\sin^2\frac{A}{2}=$$

$$=\frac{1}{4}\cdot2p\cdot\sum\sin^2\frac{A}{2}-\frac{1}{4}\cdot\sum a\sin^2\frac{A}{2}=\frac{p}{2}\left(1-\frac{r}{2R}\right)-\frac{1}{4}\cdot p\left(1-\frac{r}{R}\right)=\frac{p}{4}=\frac{a+b+c}{8}=M_d.$$

b) $\sum\frac{bc}{b+c}\cos^2\frac{A}{2}\le\frac{3(a+b+c)}{8}$**;**

IneMath 10/2016

Soluţie:

Folosim identităţile $\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}$ şi $\sum a\cos^2\frac{A}{2}=p\left(1+\frac{r}{R}\right)$. Obţinem

$$M_s=\sum\frac{bc}{b+c}\cos^2\frac{A}{2}\le\sum\frac{b+c}{4}\cos^2\frac{A}{2}=\frac{1}{4}\sum(b+c)\cos^2\frac{A}{2}=\frac{1}{4}\sum(2p-a)\cos^2\frac{A}{2}=$$

$$=\frac{1}{4}\cdot2p\cdot\sum\cos^2\frac{A}{2}-\frac{1}{4}\cdot\sum a\cos^2\frac{A}{2}=\frac{p}{2}\left(2+\frac{r}{2R}\right)-\frac{1}{4}\cdot p\left(1+\frac{r}{R}\right)=\frac{3p}{4}=M_d.$$

2. a) $\sum\frac{bc}{b+c}\mathrm{tg}^2\frac{A}{2}\le\frac{4R^2-5Rr+3r^2}{p}$**;**

Soluţie:

Folosim identităţile $\sum \operatorname{tg}^2\frac{A}{2}=\frac{(4R+r)^2-2p^2}{p^2}$ şi $\sum a\operatorname{tg}^2\frac{A}{2}=\frac{4R(4R+r)-2p^2}{p}$. Obţinem

$$M_s=\sum\frac{bc}{b+c}\operatorname{tg}^2\frac{A}{2}\le\sum\frac{b+c}{4}\operatorname{tg}^2\frac{A}{2}=\frac{1}{4}\sum(2p-a)\operatorname{tg}^2\frac{A}{2}=\frac{1}{4}\left[2p\cdot\sum\operatorname{tg}^2\frac{A}{2}-\sum a\operatorname{tg}^2\frac{A}{2}\right]=$$

$$=\frac{1}{4}\left[2p\cdot\frac{(4R+r)-2p^2}{p^2}-\frac{4R(4R+r)-2p^2}{p}\right]=\frac{1}{4p}\cdot2\left[(4R+r)^2-p^2-2R(4R+r)\right]=$$

$$=\frac{1}{2p}\left[(4R+r)(2R+r)-p^2\right]\overset{\text{Gerretsen}}{\le}\frac{1}{2p}\left[8R^2+6Rr+r^2-16Rr+5r^2\right]=\frac{1}{p}\left(4R^2-5Rr+3r^2\right)=M_d\,.$$

b) $\sum\frac{bc}{b+c}\operatorname{ctg}^2\frac{A}{2}\le\frac{p\left(2R^2-3Rr+r^2\right)}{r^2}$.

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Soluţie:

Folosim identităţile $\sum\operatorname{ctg}^2\frac{A}{2}=\frac{p^2-2r^2-8Rr}{r^2}$ şi $\sum a\operatorname{ctg}^2\frac{A}{2}=\frac{2p}{p}(2R-r)$. Obţinem

$$M_s=\sum\frac{bc}{b+c}\operatorname{ctg}^2\frac{A}{2}\le\sum\frac{b+c}{4}\operatorname{ctg}^2\frac{A}{2}=\frac{1}{4}\sum(2p-a)\operatorname{ctg}^2\frac{A}{2}=\frac{1}{4}\left[2p\cdot\sum\operatorname{ctg}^2\frac{A}{2}-\sum a\operatorname{ctg}^2\frac{A}{2}\right]=$$

$$=\frac{1}{4}\left[2p\cdot\frac{p^2-r^2-8Rr}{r^2}-\frac{2p}{r}(2R-r)\right]=\frac{2p}{4}\left[\frac{p^2-2r^2-8Rr+r^2-2Rr}{r^2}\right]=\frac{p}{2}\left[\frac{p^2-r^2-10Rr}{r^2}\right]\overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le}\frac{p}{2r^2}\left[4R^2+4Rr+3r^2-r^2-10Rr\right]=\frac{p}{r^2}\left(2R^2-3Rr+r^2\right)=M_d\,.$$

3. a) $\sum\frac{bc}{b+c}\operatorname{tg}\frac{A}{2}\le\frac{R^2-Rr+r^2}{r}$**;**

Soluţie:

$$\sum\frac{bc}{b+c}\operatorname{tg}\frac{A}{2}\le\sum\frac{b+c}{4}\operatorname{tg}\frac{A}{2}=\frac{1}{4}\sum(2p-a)\operatorname{tg}\frac{A}{2}=\frac{1}{4}\left[2p\cdot\sum\operatorname{tg}\frac{A}{2}-\sum a\operatorname{tg}\frac{A}{2}\right]=$$

$$=\frac{1}{4}\left[2p\cdot\frac{a^2+b^2+c^2}{4S}-2(2R-r)\right]=\frac{1}{4}\left[p\cdot\frac{2(p^2-r^2-4Rr}{2rp}-2(2R-r)\right]=\frac{1}{4r}\left(p^2+r^2-8Rr\right)\overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le}\frac{1}{4r}\left(4R^2+4Rr+3r^2+r^2-8Rr\right)=\frac{1}{r}R^2-Rr+r^2\,.$$

b) $\sum\frac{bc}{b+c}\operatorname{tg}\frac{A}{2}\le R+r$**;**

Soluţie:

$$\sum\frac{bc}{b+c}\operatorname{tg}\frac{A}{2}\le\frac{1}{4}\left[2p\cdot\sum\operatorname{tg}\frac{A}{2}-\sum a\operatorname{tg}\frac{A}{2}\right]=\frac{1}{4}\left[2p\cdot\frac{4R+r}{p}-2(2R-r)\right]=\frac{1}{2}\left(2R+2r\right)=R+r\,.$$

c) $\sum\frac{bc}{b+c}\operatorname{tg}\frac{A}{2}\le R+r\le\frac{R^2-Rr+r^2}{r}$**;**

Soluţie:

Vezi b) şi inegalitatea lui Euler.

d) $\sum \frac{bc}{b+c}\operatorname{ctg}\frac{A}{2} \le \frac{2R^2+r^2}{r}$.

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Soluţie: $M_s \le \sum \frac{b+c}{4}\operatorname{ctg}\frac{A}{2} = \frac{1}{4}\sum(2p-a)\operatorname{ctg}\frac{A}{2} = \frac{1}{4}\left[2p\cdot\sum\operatorname{ctg}\frac{A}{2} - \sum a\operatorname{ctg}\frac{A}{2}\right] = \frac{1}{4}\left[2p\cdot\frac{p}{r} - 2(4R+r)\right] =$

$= \frac{1}{2r}\left[p^2 - r(4R+r)\right] \overset{\text{Gerretsen}}{\le} \frac{1}{2r}\left(4R^2+4Rr+3r^2-4Rr-r^2\right) = \frac{1}{r}\left(2R^2+r^2\right) = M_d$.

4. a) $\sum \frac{bc}{b+c}\sin A \le \frac{(R+r)^2}{R}$;

Soluţie:

$M_s \le \sum \frac{b+c}{4}\sin A = \frac{1}{4}\sum(2p-a)\sin A = \frac{1}{4}\left[2p\cdot\sum\sin A - \sum a\sin A\right] = \frac{1}{4}\left[2p\cdot\frac{p}{R} -\right.$

$\left.-\frac{p^2-r^2-4Rr}{R}\right] = \frac{1}{4R}\left[p^2+r^2+4Rr\right] \overset{\text{Gerretsen}}{\le} \frac{1}{4R}\left(4R^2+4Rr+3r^2+r^2+4Rr\right) = \frac{1}{R}\left(R^2+2Rr+\right.$

$\left.+r^2\right) = \frac{1}{R}(R+r)^2 = M_d$.

b) $\sum \frac{bc}{b+c}\cos A \le \frac{p}{2}$.

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Soluţie:

$M_s \le \sum \frac{b+c}{4}\cos A = \frac{1}{4}\sum(2p-a)\cos A = \frac{1}{4}\left[2p\cdot\sum\cos A - \sum a\cos A\right] =$

$= \frac{1}{4}\left[2p\cdot\left(1+\frac{r}{R}\right) - \frac{2rp}{R}\right] = \frac{2p}{4R}\left[R+r-r\right] = \frac{p}{2} = M_d$.

5. a) $\sum \frac{bc}{b+c}\sin B\sin C \le \frac{p\left(2R^2+Rr+2r^2\right)}{4R^2}$;

Soluţie:

$M_s \le \sum \frac{b+c}{4}\sin B\sin C = \frac{1}{4}\sum(2p-a)\sin B\sin C = \frac{1}{4}\left[2p\cdot\sum\sin B\sin C -\right.$

$\left.-\sum a\sin B\sin C\right] = \frac{1}{4}\left[2p\cdot\frac{p^2+r^2+4Rr}{4R^2} - \frac{3rp}{R}\right] = \frac{p}{8R^2}\left[p^2+r^2-2Rr\right] \overset{\text{Gerretsen}}{\le} \frac{p}{8R^2}\left(4R^2+4Rr+\right.$

$\left.+3r^2+r^2-2Rr\right) = \frac{p}{4R^2}\left(2R^2+Rr+2r^2\right) = M_d$.

b) $\sum \frac{bc}{b+c}\cos B\cos C \le \frac{pr(R+2r)}{4R^2}$.

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Soluţie:

$$M_s \le \sum \frac{b+c}{4}\cos B\cos C = \frac{1}{4}\sum (2p-a)\cos B\cos C = \frac{1}{4}\Big[2p\cdot\sum \cos B\cos C -$$

$$-\sum a\cos B\cos C\Big] = \frac{1}{4}\left[2p\cdot\frac{p^2+r^2-4R^2}{4R^2}-\frac{rp}{R}\right] = \frac{p}{8R^2}\left[p^2+r^2-4R^2-2Rr\right] \overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le} \frac{p}{8R^2}\left(4R^2+4Rr+3r^2+r^2-4R^2-2Rr\right) = \frac{rp}{4R^2}(R+2r) = M_d .$$

6. **a)** $\sum \frac{bc}{b+c}(\sin B+\sin C) \le \frac{3R^2+2Rr+2r^2}{R}$;

Soluţie:

$$M_s \le \sum \frac{b+c}{4}(\sin B+\sin C) = \frac{1}{4}\sum (2p-a)(\sin B+\sin C) = \frac{1}{4}\Big[2p\cdot\sum (\sin B+\sin C) -$$

$$-\sum a(\sin B+\sin C)\Big] = \frac{1}{4}\left[2p\cdot\frac{2p}{R}-\frac{p^2+r^2+4Rr}{R}\right] = \frac{1}{4}\cdot\frac{4p^2-p^2-r^2+4Rr}{R} \overset{\text{Gerretsen}}{\le} \frac{1}{4R}\Big[3\big(4R^2+$$

$$+4Rr+3r^2\big)-r^2-4Rr\Big] = \frac{1}{R}\left(3R^2+2Rr+2r^2\right) = M_d .$$

b) $\sum \frac{bc}{b+c}(\cos B+\cos C) \le \frac{p(R+2r)}{2R}$.

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Soluţie:

$$M_s \le \sum \frac{b+c}{4}(\cos B+\cos C) = \frac{1}{4}\sum (2p-a)(\cos B+\cos C) = \frac{1}{4}\Big[2p\cdot\sum (\cos B+\cos C) -$$

$$-\sum a(\cos B+\cos C)\Big] = \frac{1}{4}\left[2p\cdot 2\left(1+\frac{r}{R}\right)-2p\right] = \frac{p}{2}\left(1+\frac{2r}{R}\right) = \frac{p}{2R}(R+2r) = M_d .$$

7. **a)** $\sum \frac{bc}{b+c}\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right) \le \frac{p(3R-2r)}{4R}$;

Soluţie: $M_s \le \sum \frac{b+c}{4}\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right) = \frac{1}{4}\sum (2p-a)\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right) = \frac{1}{4}\Big[2p\cdot\sum\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right) -$

$$-\sum a\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\Big] = \frac{1}{4}\left[2p\cdot 2\left(1-\frac{r}{2R}\right)-p\right] = \frac{p}{4}\cdot\left(4-\frac{2r}{R}-1\right) = \frac{p}{4R}(3R-2r) = M_d .$$

b) $\sum \frac{bc}{b+c}\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\le \frac{p\left(5R+2r\right)}{4R}$.

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Soluţie:

$$M_s \le \sum \frac{b+c}{4}\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)=\frac{1}{4}\sum(2p-a)\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)=$$

$$=\frac{1}{4}\left[2p\cdot\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)-\sum a\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\right]=\frac{1}{4}\left[2p\cdot 2\left(2+\frac{r}{2R}\right)-3p\right]=$$

$$=\frac{p}{4}\left(5+\frac{2r}{R}\right)=\frac{p}{4R}\left(5R+2r\right)=M_d .$$

8. **a)** $\sum \frac{bc}{b+c}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\le \frac{p\left(2R^2-3Rr+2r^2\right)}{16R^2}$**;**

Soluţie:

$$M_s \le \sum \frac{b+c}{4}\sin^2\frac{B}{2}\sin^2\frac{C}{2}=\frac{1}{4}\sum(2p-a)\sin^2\frac{B}{2}\sin^2\frac{C}{2}=\frac{1}{4}\left[2p\cdot\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2}-\right.$$

$$\left.-\sum a\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right]=\frac{1}{4}\left[2p\cdot\frac{p^2+r^2-8Rr}{16R^2}-\frac{rp}{4R}\right]=\frac{p}{4}\cdot\frac{p^2+r^2-10Rr}{8R^2}\overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le}\frac{p}{4}\cdot\frac{4R^2+4Rr+3r^2+r^2-10Rr}{8R^2}=\frac{p}{16R^2}\left(2R^2-3Rr+2r^2\right)=M_d .$$

b) $\sum \frac{bc}{b+c}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\le \frac{p\left(6R^2+5Rr+2r^2\right)}{16R^2}$

Soluţie:

$$M_s \le \sum \frac{b+c}{4}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=\frac{1}{4}\sum(2p-a)\cos^2\frac{B}{2}\cos^2\frac{C}{2}=\frac{1}{4}\left[2p\cdot\sum\cos^2\frac{B}{2}\cos^2\frac{C}{2}-\right.$$

$$\left.-\sum a\cos^2\frac{B}{2}\cos^2\frac{C}{2}\right]=\frac{1}{4}\left[2p\cdot\frac{p^2+(4R+r)^2}{16R^2}-\frac{p(4R+r)}{4R}\right]=\frac{p}{4}\cdot\frac{p^2+(4R+r)^2-2R(4R+r)}{8R^2}\overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le}\frac{p}{4}\cdot\frac{12R^2+10Rr+4r^2}{8R^2}=\frac{p}{16R^2}\left(6R^2+5Rr+2r^2\right)=M_d .$$

c) $\sum \frac{bc}{b+c}\mathrm{tg}^2\frac{B}{2}\mathrm{tg}^2\frac{C}{2}\le \frac{2R^2-3Rr+r^2}{p}$.

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Soluţie:

$$M_s \le \sum \frac{b+c}{4}\operatorname{tg}^2\frac{B}{2}\operatorname{tg}^2\frac{C}{2} = \frac{1}{4}\sum (2p-a)\operatorname{tg}^2\frac{B}{2}\operatorname{tg}^2\frac{C}{2} = \frac{1}{4}\left[2p\cdot\sum \operatorname{tg}^2\frac{B}{2}\operatorname{tg}^2\frac{C}{2} -\right.$$

$$\left.-\sum a\operatorname{tg}^2\frac{B}{2}\operatorname{tg}^2\frac{C}{2}\right] = \frac{1}{4}\left[2p\cdot\frac{p^2-2r^2-8Rr}{16R^2} - \frac{2r}{p}(2R-r)\right] = \frac{1}{2p}\cdot\left[2p^2-2r^2-8Rr-2Rr+r^2\right] =$$

$$= \frac{1}{2p}\left(p^2-r^2-10Rr\right) \overset{\text{Gerretsen}}{\le} \frac{1}{2p}(4R^2+4Rr+3r^2 - r^2 - 10Rr = \frac{1}{p}\left(2R^2-3Rr+r^2\right) = M_d\,.$$

9. a) $\sum \frac{bc}{b+c}\sin^2 A \le \frac{p\left(2R^2+Rr+2r^2\right)}{4R^2}$**;**

Soluţie: $M_s \le \sum \frac{b+c}{4}\sin^2 A = \frac{1}{4}\sum (b+c)\sin^2 A = \frac{1}{4}\cdot\frac{p\left(p^2+r^2-2Rr\right)}{2R^2} \overset{\text{Gerretsen}}{\le} \frac{p}{8R^2}\left(4R^2+4Rr+\right.$

$\left.+3r^2+r^2-2Rr\right) = \frac{p}{4R^2}\left(2R^2+Rr+2r^2\right) = M_d\,.$

b) $\sum \frac{bc}{b+c}\cos^2 A \le \frac{p\left(4R^2-7Rr+2r^2\right)}{4R^2}$**.**

Dezvoltări, M.Chirciu

Soluţie:

$$M_s \le \sum \frac{b+c}{4}\cos^2 A = \frac{1}{4}\sum (b+c)\cos^2 A = \frac{1}{4}\frac{p\left(8R^2+2Rr-r^2-p^2\right)}{2R^2} \overset{\text{Gerretsen}}{\le}$$

$$\overset{\text{Gerretsen}}{\le} \frac{p}{8R^2}\left(8R^2+2Rr-r^2-16Rr+5r^2\right) = \frac{p}{4R^2}\left(4R^2-7Rr+2r^2\right) = M_d\,.$$

La fiecare dintre inegalitățile de mai sus egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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