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ARTICOLE REVISTĂ:

1. PROBLEMA LUNII DECEMBRIE 2017

a) Sa se arate ca pentru orice $x \in \mathbb{R}$ exista un triunghi ABC avand lungimile laturilor:

$$c = \mathrm{AB} = \sqrt{2x^2 - 2x + 3},\ \mathrm{b{=}AC{=}}\sqrt{2x^2 + 2x + 3},\ \mathrm{a{=}BC{=}}\sqrt{8x^2 + 10}\ .$$

b) Aria triunghiului ABC este un numar irational care nu depinde de x.

Propusa de : Prof. Iuliana Trasca

2. SOLUȚII PROBLEMA LUNII NOIEMBRIE 2017

Determinați toate numerele reale pozitive a și b pentru care

$$\frac{ab}{ab+n}+\frac{a^2b}{a^2+nb}+\frac{ab^2}{b^2+na}=\frac{1}{n+1}(a+b+ab),\text{ unde } n\in \boldsymbol{N}^* .$$

Marin Chirciu, Pitesti

Soluție.

Notând $a=\frac{1}{x},b=\frac{1}{y}$ egalitatea se scrie:

$$\frac{1}{1+nxy}+\frac{1}{x^2+ny}+\frac{1}{y^2+nx}=\frac{1+x+y}{(n+1)xy},\ (1).$$

Folosind inegalitatea lui Bergström obținem:

$$\frac{1}{1+nxy}+\frac{1}{x^2+ny}+\frac{1}{y^2+nx}\geq\frac{(1+1+1)^2}{1+nxy+x^2+ny+y^2+nx}=\frac{9}{1+nx+ny+nxy+x^2+y^2}(2).$$

Din (1) și (2) rezultă:

$$\frac{1+x+y}{(n+1)xy}\geq\frac{9}{1+nx+ny+nxy+x^2+y^2}\Leftrightarrow(1+x+y)(1+nx+ny+nxy+x^2+y^2)\geq 9(n+1)xy.$$

Din inegalitatea mediilor avem:

$$1+x+y\geq 3\sqrt[3]{xy}\ \text{ și }\ 1+nx+ny+nxy+x^2+y^2\geq 3(n+1)\sqrt[2(n+1)]{(xy)^{2(n+1)}}=3(n+1)\sqrt[3]{(xy)^2}.$$

Deducem că: $(1+x+y)(1+nx+ny+nxy+x^2+y^2)\geq 3\sqrt[3]{xy}\cdot 3(n+1)\sqrt[3]{(xy)^2}=9(n+1)xy$

cu egalitatea dacă și numai dacă $x=y=1$.

Egalitatea din enunț are loc dacă și numai dacă $a=b=1$.

Notă.

Pentru $n=1$ se obține problema J.416 din Mathematical Reflections 4/2017, autor Mihaela Berindeanu, București.

SOLUȚIE DATĂ DE BIRO ISTVAN

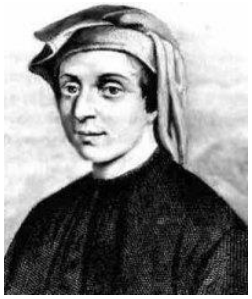
Observăm că $a=0\Rightarrow b=0$ și $b=0\Rightarrow a=0$, adică a și b sunt strict pozitive. Ecuația din enunț se poate scrie ca o ecuație polinomială de gradul 3 cu necunoscuta n și coeficienți reali ce depind de a și b. De fapt este vorba de o identitate valabilă pentru orice n nenul, de unde rezultă că se caută a și b pentru care coeficienții devin simultan nuli, adică:

$$\begin{cases} a^3b^3(1-ab)=0 & (0)\ \textit{termenul liber} \\ ab\left[(1-ab)(a^3+b^3)+a^3(b^2-a)+b^3(a^2-b)\right]=0 & (1)\ \textit{coeficientul lui } n \\ a^2b^2(1-ab)+a^4(b^2-1)+b^4(a^2-1)=0 & (2)\ \textit{coeficientul lui } n^2 \\ ab(a^2+b^2-a-b)=0 & (3)\ \textit{coeficientul lui } n^3 \end{cases}$$

Din (0) și (3) rezultă cu ușurință că singura soluție este $a=b=1$.

3. THE NUMBERS of FIBONACCI and LUCAS - IDENTITIES - PROOFS WITH FEW WORDS – (II)

By Dumitru M. Bătineţu-Giurgiu, Bucharest, Romania
and Neculai Stanciu, Buzău, Romania



Fibonacci

(1175 -1240)

François-Édouard-Anatole Lucas

(1842 – 1891)

$$F_0 = 0, F_1 = 1,$$
$$F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbf{N} \qquad \text{(F)}$$
$$L_0 = 2, L_1 = 1,$$
$$L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbf{N} \qquad \text{(L)}$$
$$r^2 - r - 1 = 0,$$
$$r_1 = \alpha = \frac{1+\sqrt{5}}{2}, r_2 = \beta = \frac{1-\sqrt{5}}{2}.$$

$(x_n)_{n\geq 0}$, **Fibonacci-Lucas**' s sequence

$$x_n = A\alpha^n + B\beta^n,\ \forall n \in \mathbf{N},$$

If $x_0 = 0 = F_0, x_1 = 1 = F_1$, then $A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}}$ so:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right),\ \forall n \in \mathbf{N} \text{ (\textbf{Binet}, 1843)},$$

If $x_0 = 2 = L_0, x_1 = 1 = L_1$, then $A = B = 1$, so

$$L_n = \alpha^n + \beta^n,\ \forall n \in \mathbf{N}.$$

Note that:

$$\alpha + \beta = 1 \text{ and } \alpha\beta = -1,$$

*

* *

1.28. $L_m L_n + 5F_m F_n = 2L_{m+n}$, $\forall m,n \in \mathbf{N}$ (**Ferns**, 1967).

Proof.
$$L_m L_n + 5F_m F_n = (\alpha^m + \beta^m)(\alpha^n + \beta^n) + (\alpha^m - \beta^m)(\alpha^n - \beta^n) =$$
$$= \alpha^{m+n} + \alpha^m \beta^n + \alpha^n \beta^m + \beta^{m+n} + \alpha^{m+n} - \alpha^m \beta^n - \alpha^n \beta^m + \beta^{m+n} =$$
$$= 2(\alpha^{m+n} + \beta^{m+n}) = 2L_{m+n}.$$

1.29. $F_{n+2}^2 - F_{n+1}^2 = F_n F_{n+3}$, $\forall n \in \mathbf{N}$.

Proof. $F_{n+2}^2 - F_{n+1}^2 = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = F_n F_{n+3}$.

1.30. $L_{m+n}^2 - L_{m-n}^2 = 5F_{2m}F_{2n}$, $\forall m,n \in \mathbf{N}, m \geq n$ (**Koshy**, 1998).

Proof.
$$L_{m+n}^2 - L_{m-n}^2 = (\alpha^{m+n} + \beta^{m+n})^2 - (\alpha^{m-n} + \beta^{m-n})^2 =$$
$$= \alpha^{2m+2n} + \beta^{2m+2n} + 2(\alpha\beta)^{m+n} - \alpha^{2m-2n} - \beta^{2m-2n} - 2(\alpha\beta)^{m-n} =$$
$$= \alpha^{2m}(\alpha^{2n} - \alpha^{-2n}) + \beta^{2m}(\beta^{2n} - \beta^{-2n}) + 2(\alpha\beta)^{m-n}\left((\alpha\beta)^{2n} - 1\right) =$$
$$= \alpha^{2m}(\alpha^{2n} - \alpha^{-2n}) + \beta^{2m}(\beta^{2n} - \beta^{-2n}) = 5F_{2m}F_{2n}.$$

1.31. $L_{m+n}^2 + L_{m-n}^2 = L_{2m}L_{2n} + 4(-1)^{m+n}$, $\forall m,n \in \mathbf{N}, m \geq n$ (**Koshy**, 1998).

Proof.
$$L_{m+n}^2 + L_{m-n}^2 = (\alpha^{m+n} + \beta^{m+n})^2 - (\alpha^{m-n} + \beta^{m-n})^2 =$$
$$= \alpha^{2m+2n} + \beta^{2m+2n} + 2(\alpha\beta)^{m+n} + \alpha^{2m-2n} + \beta^{2m-2n} + 2(\alpha\beta)^{m-n} =$$
$$= \alpha^{2m}(\alpha^{2n} + \alpha^{-2n}) + \beta^{2m}(\beta^{2n} + \beta^{-2n}) + 2(\alpha\beta)^{m-n}\left((\alpha\beta)^{-2n} + 1\right) =$$
$$= (\alpha^{2m} + \beta^{2m})(\alpha^{2n} + \beta^{2n}) + 4(\alpha\beta)^{m+n} = L_{2m}L_{2n} + 4(-1)^{m+n}.$$

1.32. $5(F_{m+n}^2 + F_{m-n}^2) = L_{2m}L_{2n} - 4(-1)^{m+n}$, $\forall m,n \in \mathbf{N}, m \geq n$ (**Koshy**, 1999).

Proof.
$$5(F_{m+n}^2 + F_{m-n}^2) = 5\left(\frac{1}{5}(\alpha^{n+m} + \beta^{n+m})^2 + \frac{1}{5}(\alpha^{m-n} - \beta^{m-n})\right)^2 =$$
$$= \alpha^{2m+2n} + \beta^{2m+2n} - 2(\alpha\beta)^{m+n} + \alpha^{2m-2n} + \beta^{2m-2n} - 2(\alpha\beta)^{m-n} =$$
$$= \alpha^{2m}(\alpha^{2n} + \alpha^{-2n}) + \beta^{2m}(\beta^{2n} + \beta^{-2n}) - 2(\alpha\beta)^{m+n}\left(1 + (\alpha\beta)^{-2n}\right) =$$
$$= \alpha^{2m}(\alpha^{2n} + \beta^{2n}) + \beta^{2m}(\beta^{2n} + \alpha^{2n}) - 4(\alpha\beta)^{m+n} =$$
$$= (\alpha^{2m} + \beta^{2m})(\alpha^{2n} + \beta^{2n}) - 4(-1)^{m+n} = L_{2m}L_{2n} - 4(-1)^{m+n}.$$

1.33. $1 + F_{2n}F_{2n+2} = F_{2n+1}^2$, $\forall n \in \mathbf{N}$ (**Cassini**).

Proof. $1+F_{2n}F_{2n+2}=1+\frac{1}{5}(\alpha^{2n}-\beta^{2n})(\alpha^{2n+2}-\beta^{2n+2})=$

$$=1+\frac{1}{5}\left(\alpha^{4n+2}-\alpha^{2n}\beta^{2n+2}-\alpha^{2n+2}\beta^{2n}+\beta^{4n+2}\right)=$$

$$=1+\frac{1}{5}\left(\alpha^{4n+2}+\beta^{4n+2}-2\alpha^{2n+1}\beta^{2n+1}+2\alpha^{2n+1}\beta^{2n+1}-\alpha^{2n}\beta^{2n+2}-\alpha^{2n+2}\beta^{2n}\right)=$$

$$=1+\frac{1}{5}(\alpha^{2n+1}-\beta^{2n+1})^2+\frac{1}{5}\left(2(\alpha\beta)^{2n+1}-(\alpha\beta)^{2n}(\alpha^2+\beta^2)\right)=$$

$$=F_{2n+1}^2+1+\frac{1}{5}(-2-\alpha^2-\beta^2)=F_{2n+1}^2+1-\frac{1}{5}(2+3)=F_{2n+1}^2.$$

1.34. $F_{2n+1}F_{2n+3}-1=F_{2n+2}^2$, $\forall n\in\mathbf{N}$ (**Cassini**).

Proof. $F_{2n+1}F_{2n+3}-1=-1+\frac{1}{5}(\alpha^{2n+1}-\beta^{2n+1})(\alpha^{2n+3}-\beta^{2n+3})=$

$$=-1+\frac{1}{5}\left(\alpha^{4n+4}+\beta^{4n+4}-2(\alpha\beta)^{2n+2}+2(\alpha\beta)^{2n+2}-\alpha^{2n+1}\beta^{2n+3}-\alpha^{2n+3}\beta^{2n+1}\right)=$$

$$=-1+\frac{1}{5}(\alpha^{2n+2}-\beta^{2n+2})^2+2-(\alpha\beta)^{2n+1}(\alpha^2+\beta^2)=$$

$$=F_{2n+2}^2-1+\frac{1}{5}(2+\alpha^2+\beta^2)=F_{2n+2}^2-1+\frac{1}{5}\left(2+\frac{6+2\sqrt{5}}{4}+\frac{6-2\sqrt{5}}{4}\right)=$$

$$=F_{2n+2}^2-1+\frac{1}{5}(2+3)=F_{2n+2}^2-1+1=F_{2n+2}^2.$$

1.35. $1+F_{2n+2}F_{2n+4}=F_{2n+3}^2$, $\forall n\in\mathbf{N}$ (**Cassini**).

Proof. $1+F_{2n+2}F_{2n+4}=1+\frac{1}{5}(\alpha^{2n+2}-\beta^{2n+2})(\alpha^{2n+4}-\beta^{2n+4})=$

$$=1+\frac{1}{5}(\alpha^{4n+6}+\beta^{4n+6}-2(\alpha\beta)^{2n+3}+2(\alpha\beta)^{2n+3}-\alpha^{2n+2}\beta^{2n+4}-\alpha^{2n+4}\beta^{2n+2})=$$

$$=1+\frac{1}{5}(\alpha^{2n+3}-\beta^{2n+3})^2+\frac{1}{5}\left(2(-1)^{2n+3}-(-1)^{2n+2}\beta^2-(-1)^{2n+2}\alpha^2\right)=$$

$$=F_{2n+3}^2+1-\frac{1}{5}(2+\alpha^2+\beta^2)=F_{2n+3}^2+1-1=F_{2n+3}^2.$$

1.36. $4F_{2n}F_{2n+1}F_{2n+2}F_{2n+3}+1=(2F_{2n+1}F_{2n+2}-1)^2$, $\forall n\in\mathbf{N}$.

Proof. $4F_{2n}F_{2n+1}F_{2n+2}F_{2n+3}+1=1+4(F_{2n}F_{2n+2})(F_{2n+1}F_{2n+3})\overset{\text{Cassini}}{=}$

$$=1+4(F_{2n+1}^2-1)(F_{2n+2}^2+1)=4F_{2n+1}^2F_{2n+2}^2-4(F_{2n+2}^2-F_{2n+1}^2)-3=$$

$$=4F_{2n+1}^2F_{2n+2}^2-4(F_{2n+2}-F_{2n+1})(F_{2n+2}+F_{2n+1})-3=$$

$$=4F_{2n+1}^2F_{2n+2}^2-4F_{2n}F_{2n+3}-3=4F_{2n+1}^2F_{2n+2}^2-4F_{2n+3}(F_{2n+2}-F_{2n+1})-3=$$
$$=4F_{2n+1}^2F_{2n+2}^2-4F_{2n+2}F_{2n+3}+4F_{2n+1}F_{2n+3}-3=$$
$$=4F_{2n+1}^2F_{2n+2}^2-4F_{2n+2}F_{2n+3}+4(F_{2n+2}^2+1)-3=$$
$$=4F_{2n+1}^2F_{2n+2}^2-4F_{2n+2}(F_{2n+3}-F_{2n+2})+1=$$
$$=4F_{2n+1}^2F_{2n+2}^2-4F_{2n+1}F_{2n+2}+1=(2F_{2n+1}F_{2n+2}-1)^2.$$

1.37. $4F_{2n+1}F_{2n+2}^2F_{2n+3}+1=(2F_{2n+2}^2+1)^2$, $\forall n\in\mathbf{N}$.

Proof. $4F_{2n+1}F_{2n+2}^2F_{2n+3}+1=1+4F_{2n+2}^2(F_{2n+1}F_{2n+3})\overset{\text{Cassini}}{=}$
$$=1+4F_{2n+2}^2(F_{2n+2}^2+1)=4F_{2n+2}^4+4F_{2n+2}^2+1=(2F_{2n+2}^2+1)^2.$$

1.38. $F_{n+2}^2-F_n^2=F_{2n+2}$, $\forall n\in\mathbf{N}$.

Proof. $F_{n+2}^2-F_n^2=\frac{1}{5}\left(\alpha^{n+2}-\beta^{n+2}\right)^2-\frac{1}{5}\left(\alpha^n-\beta^n\right)^2=$
$$=\frac{1}{5}\left(\alpha^{2n+4}+\beta^{2n+4}-2(\alpha\beta)^{n+2}-\alpha^{2n}-\beta^{2n}+2(\alpha\beta)^n\right)=$$
$$=\frac{1}{5}\left(\alpha^{2n+2}(\alpha^2-\alpha^{-2})+\beta^{2n+2}(\beta^2-\beta^{-2})+2(\alpha\beta)^n(1-(\alpha\beta)^2)\right)=$$
$$=\frac{1}{5}\left(\alpha^{2n+2}(\alpha^2-\beta^2)+\beta^{2n+2}(\beta^2-\alpha^2)\right)=\frac{1}{5}(\alpha^2-\beta^2)(\alpha^{2n+2}-\beta^{2n+2})=$$
$$=\frac{1}{5}(\alpha-\beta)(\alpha+\beta)(\alpha^{2n+2}-\beta^{2n+2})=\frac{1}{5}\cdot\sqrt{5}\cdot1\cdot(\alpha^{2n+2}-\beta^{2n+2})=$$
$$=\frac{1}{\sqrt{5}}(\alpha^{2n+2}-\beta^{2n+2})=F_{2n+2}.$$

1.39. $F_m\cdot L_n+F_n\cdot L_m=2F_{m+n}$, $\forall m,n\in\mathbf{N}$ (**Ferns**, 1967).

Proof. $F_m\cdot L_n+F_n\cdot L_m=\frac{1}{\sqrt{5}}(\alpha^m-\beta^m)(\alpha^n+\beta^n)+\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)(\alpha^m+\beta^m)=$
$$=\frac{1}{\sqrt{5}}(\alpha^{m+n}-\alpha^n\beta^m+\alpha^m\beta^n-\beta^{m+n}+\alpha^{m+n}-\alpha^m\beta^n+\alpha^n\beta^m-\beta^{m+n})=$$
$$=\frac{2}{\sqrt{5}}(\alpha^{m+n}-\beta^{m+n})=2F_{m+n}.$$

1.40. $L_n^2+L_{n+1}^2=5F_{2n+1}$, $\forall n\in\mathbf{N}$.

Proof.. $L_n^2+L_{n+1}^2=(\alpha^n+\beta^n)^2+(\alpha^{n+1}+\beta^{n+1})^2=$
$$=\alpha^{2n}+\beta^{2n}+2(\alpha\beta)^n+\alpha^{2n+2}+\beta^{2n+2}+2(\alpha\beta)^{n+1}=\alpha^{2n+1}(\alpha^{-1}+\alpha)+$$

$$+\beta^{2n+1}(\beta+\beta^{-1})+2(\alpha\beta)^{n+1}(1+\alpha\beta)=(\alpha-\beta)(\alpha^{2n+1}-\beta^{2n+1})=$$
$$=\sqrt{5}(\alpha^{2n+1}-\beta^{2n+1})=5\cdot\frac{1}{\sqrt{5}}(\alpha^{2n+1}-\beta^{2n+1})=5F_{2n+1}.$$

1.41. $L_nL_{n+2}-L_{n+1}^2=5(-1)^n$, $\forall n\in\mathbf{N}$.

Proof. $L_nL_{n+2}-L_{n+1}^2=(\alpha^n+\beta^n)(\alpha^{n+2}+\beta^{n+2})-(\alpha^{n+1}+\beta^{n+1})^2=$
$$=\alpha^{2n+2}+\alpha^{n+2}\beta^n+\alpha^n\beta^{n+2}+\beta^{2n+2}-\alpha^{2n+2}-\beta^{2n+2}-2(\alpha\beta)^{n+1}=$$
$$=(\alpha\beta)^n(\alpha^2+\beta^2)-2(\alpha\beta)^{n+1}=(\alpha\beta)^n(\alpha^2+\beta^2-2\alpha\beta)=$$
$$=(-1)^n(\alpha-\beta)^2=(-1)^n\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right)^2=(-1)^n(\sqrt{5})^2=5(-1)^n.$$

1.42. $5F_n^2=L_n^2-4(-1)^n$, $\forall n\in\mathbf{N}$.

Proof. $5F_n^2=5\cdot\left(\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)\right)^2=\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n=$
$$=(\alpha^n+\beta^n)^2-4(\alpha\beta)^n=L_n^2-4(-1)^n.$$

1.43. $L_n^2=L_{2n}+2(-1)^n$, $\forall n\in\mathbf{N}^*$.

Proof. $L_n^2=(\alpha^n+\beta^n)^2=\alpha^{2n}+\beta^{2n}+2(\alpha\beta)^n=L_{2n}+2(-1)^n$.

1.44. $L_{n+4}-L_n=5F_{n+2}$, $\forall n\in\mathbf{N}$.

Proof. $L_{n+4}-L_n=\alpha^{n+4}+\beta^{n+4}-\alpha^n-\beta^n=$
$$=\alpha^{n+2}(\alpha^2-\alpha^{-2})+\beta^{n+2}(\beta^2-\beta^{-2})=(\alpha^2-\beta^2)(\alpha^{n+2}-\beta^{n+2})=$$
$$=(\alpha-\beta)(\alpha+\beta)(\alpha^{n+2}-\beta^{n+2})=\sqrt{5}(\alpha^{n+2}-\beta^{n+2})=5\cdot\frac{1}{\sqrt{5}}(\alpha^{n+2}-\beta^{n+2})=5F_{n+2}.$$

1.45. $L_{2n}=5F_n^2+2(-1)^n$, $\forall n\in\mathbf{N}^*$.

Proof. $L_{2n}=\alpha^{2n}+\beta^{2n}=(\alpha^n-\beta^n)^2+2(\alpha\beta)^n=5\left(\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)\right)^2+$
$$+2(-1)^n=5F_n^2+2(-1)^n.$$

1.46. $L_{2m+n}-(-1)^mL_n=5F_mF_{m+n}$, $\forall m,n\in\mathbf{N}$.

Proof. $L_{2m+n}-(-1)^m L_n = \alpha^{2m+n}+\beta^{2m+n}-(-1)^m(\alpha^n+\beta^n)$

$$5F_mF_{m+n} = 5\cdot\frac{1}{\sqrt{5}}(\alpha^m-\beta^m)\frac{1}{\sqrt{5}}(\alpha^{m+n}-\beta^{m+n}) = \alpha^{2m+n}-\alpha^m\beta^{m+n}-\alpha^{m+n}\beta^m+\beta^{2m+n} =$$
$$=\alpha^{2m+n}+\beta^{2m+n}-(\alpha\beta)^m(\alpha^n+\beta^n) = L_{2m+n}-(-1)^m L_n .$$

1.47. $F_{2m+n}-(-1)^m F_n = F_mL_{m+n}$, $\forall m,n\in\mathbf{N}$.

Proof. $F_mL_{m+n} = \frac{1}{\sqrt{5}}(\alpha^m-\beta^m)(\alpha^{m+n}+\beta^{m+n}) =$
$$=\frac{1}{\sqrt{5}}\left(\alpha^{2m+n}-\alpha^{m+n}\beta^m+\alpha^m\beta^{m+n}-\beta^{2m+n}\right) = F_{2m+n}-\frac{1}{\sqrt{5}}(\alpha\beta)^m(\alpha^n-\beta^n) =$$
$$= F_{2m+n}-(-1)^m F_n .$$

1.48. $F_{2n} = F_nF_{n+1}+F_{n-1}F_n$, $\forall n\in\mathbf{N}^*$.

Proof. $F_nF_{n+1}+F_{n-1}F_n = F_n(F_{n+1}+F_{n-1}) = \frac{1}{5}(\alpha^n-\beta^n)(\alpha^{n+1}-\beta^{n+1}+\alpha^{n-1}-$
$$-\beta^{n+1}) = \frac{1}{5}(\alpha^n-\beta^n)(\alpha^n(\alpha^2+\alpha^{-2})-\beta^n(\beta^2-\beta^{-2})) =$$
$$=\frac{1}{5}(\alpha^n-\beta^n)(\alpha^n(\alpha^2-\beta^2)-\beta^n(\beta^2-\alpha^2)) =$$
$$=\frac{1}{5}(\alpha-\beta)(\alpha+\beta)(\alpha^n-\beta^n)(\alpha^n+\beta^n) = \frac{1}{\sqrt{5}}(\alpha^{2n}-\beta^{2n}) = F_{2n} .$$

1.49. $F_{2m+n}+(-1)^m F_n = F_{m+n}L_n$, $\forall m,n\in\mathbf{N}$.

Proof. $F_{m+n}L_n = \frac{1}{\sqrt{5}}(\alpha^{m+n}-\beta^{m+n})(\alpha^m+\beta^m) = .$
$$=\frac{1}{\sqrt{5}}\left(\alpha^{2m+n}-\alpha^m\beta^{m+n}+\alpha^{m+n}\beta^m-\beta^{2m+n}\right) = \frac{1}{\sqrt{5}}(\alpha^{2m+n}-\beta^{2m+n})+$$
$$+\frac{1}{\sqrt{5}}(\alpha\beta)^m(\alpha^n-\beta^n) = F_{2m}+(-1)^m F_n .$$

1.50. $F_nF_{n+1}-F_{n-1}F_{n-2} = F_{2n-1}$, $\forall n\in\mathbf{N}^*-\{1\}$ (**Lucas**, 1876).

Proof. $F_nF_{n+1}-F_{n-1}F_{n-2} = \frac{1}{5}(\alpha^n-\beta^n)(\alpha^{n+1}-\beta^{n+1})-$
$$-\frac{1}{5}(\alpha^{n-1}-\beta^{n-1})(\alpha^{n-2}-\beta^{n-2}) = \frac{1}{5}(\alpha^{2n+1}-\alpha^{n+1}\beta^n-\alpha^n\beta^{n+1}+\beta^{2n+1}-\alpha^{2n-3}+$$

$$+\alpha^{n-2}\beta^{n-1}+\alpha^{n-1}\beta^{n-2}-\beta^{2n-3})=\frac{1}{5}(\alpha^{2n+1}+\beta^{2n+1}-\alpha^{2n-3}-\beta^{2n-3}-(\alpha\beta)^n\alpha-$$

$$-(\alpha\beta)^{n-2}\beta+(\alpha\beta)^{n-2}\alpha)=\frac{1}{5}(\alpha^{2n-1}(\alpha^2-\alpha^{-2})+\beta^{2n-1}(\beta^2-\beta^{-2})-(-1)^n\alpha-$$

$$-(-1)^n\beta+(-1)^{n-2}\alpha)=\frac{1}{5}(\alpha^{2n-1}(\alpha^2-\beta^2)+\beta^{2n-1}(\beta^2-\alpha^2))=$$

$$=\frac{1}{5}(\alpha-\beta)(\alpha+\beta)(\alpha^{2n-1}-\beta^{2n-1})=\frac{1}{\sqrt{5}}(\alpha^{2n-1}-\beta^{2n-1})=F_{2n-1}.$$

1.51. $\sqrt[n]{\alpha F_n+F_{n-1}}+\sqrt[n]{F_{n+1}-\alpha F_n}=1,\ \forall n\in\mathbf{N}^*-\{1\}.$

Proof. $\alpha F_n+F_{n-1}=\frac{1}{\sqrt{5}}(\alpha^{n+1}-\alpha\beta^n+\alpha^{n-1}-\beta^{n-1})=$

$$=\frac{1}{\sqrt{5}}(\alpha^{n+1}+\beta^{n-1}+\alpha^{n-1}-\beta^{n-1})=\frac{1}{\sqrt{5}}\cdot\alpha^n(\alpha+\alpha^{-1})=$$

$$=\frac{1}{\sqrt{5}}\cdot\alpha^n(\alpha-\beta)=\alpha^n.$$

$$F_{n+1}-\alpha F_n=\frac{1}{\sqrt{5}}(\alpha^{n+1}-\beta^{n+1}-\alpha^{n+1}+\beta^n\alpha)=$$

$$=\frac{1}{\sqrt{5}}\cdot\beta^n(\alpha-\beta)=\beta^n,$$ then we use $\alpha+\beta=1$.

1.52. $\sum_{k=0}^{2n}\binom{2n}{k}(-2)^k L_k=5^n,\ \forall n\in\mathbf{N}^*.$

Proof.

$$5^n=(2\alpha-1)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}(2\alpha)^k(-1)^{2n-k}=\sum_{k=0}^{2n}\binom{2n}{k}(2\alpha)^k(-1)^k=\sum_{k=0}^{2n}\binom{2n}{k}(-2\alpha)^k \quad (1)$$

$$5^n=(1-2\beta)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}(-2\beta)^k \quad (2)$$

Adding (1) and (2) we obtain:

$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k 2^{k-1}(\alpha^k+\beta^k)=\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k 2^{k-1}L_k,$$ yields the conclusion.

1.53. $\sum_{k=0}^{n}\binom{n}{k}F_{4mk}=F_{2mn}L_{2m}^n,\quad \forall m,n\in\mathbf{N}.$

Proof. $\sqrt{5}\cdot\sum_{k=0}^{n}\binom{n}{k}F_{4mk}=\sum_{k=0}^{n}\binom{n}{k}(\alpha^{4mk}-\beta^{4mk})=\sum_{k=0}^{n}\binom{n}{k}\alpha^{4mk}-\sum_{k=0}^{n}\binom{n}{k}\beta^{4mk}=$

$$=(1+\alpha^{4m})^n-(1+\beta^{4m})^n=\alpha^{2mn}(\alpha^{2m}+\alpha^{-2m})^n-\beta^{2mn}(\beta^{2m}+\beta^{-2m})^n=$$

$$=\alpha^{2mn}(\alpha^{2m}+\beta^{2m})^n-\beta^{2mn}(\alpha^{2m}+\beta^{2m})^n=(\alpha^{2m}+\beta^{2m})^n(\alpha^{2mn}-\beta^{2mn})=$$
$=\sqrt{5}L_{2m}^nF_{2mn}$, follows the conclusion.

1.54. $L_{2m+n}-(-1)^mL_n=5F_mF_{m+n}$, $\forall m,n\in\mathbf{N}$.

Proof. $L_{2m+n}-(-1)^mL_n=\alpha^{2m+n}+\beta^{2m+n}-\alpha^m\beta^m(\alpha^n+\beta^n)=$
$$=\alpha^{m+n}(\alpha^m-\beta^m)-\beta^{m+n}(\alpha^m-\beta^m)=(\alpha^{m+n}-\beta^{m+n})(\alpha^m-\beta^m)=5F_mF_{m+n}.$$

1.55. $L_{3n}=L_nL_{2n}-(-1)^nL_n$, $\forall n\in\mathbf{N}$.

Proof. $L_{3n}=\alpha^{3n}+\beta^{3n}=(\alpha^n+\beta^n)(\alpha^{2n}-(\alpha\beta)^n+\beta^{2n})=L_n(L_{2n}-(-1)^n)$.

1.56. $L_{2n}=\sum_{k=0}^{n}\binom{n}{k}L_k$, $\forall n\in\mathbf{N}^*$.

Proof. $L_{2n}=\alpha^{2n}+\beta^{2n}=(1+\alpha)^n+(1+\beta)^n=\sum_{k=0}^{n}\binom{n}{k}\alpha^k+\sum_{k=0}^{n}\binom{n}{k}\beta^k=$
$$=\sum_{k=0}^{n}\binom{n}{k}(\alpha^k+\beta^k)=\sum_{k=0}^{n}\binom{n}{k}L_k.$$

1.57. $F_{2n+i}=\sum_{k=0}^{n}\binom{n}{k}F_{k+i}$, $\forall n\in\mathbf{N}^*,\forall i\in\mathbf{N}$.

Proof. $\sum_{k=0}^{n}\binom{n}{k}F_{k+i}=\frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\alpha^{k+i}-\beta^{k+i})=\frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\alpha^{k+i}-$
$$-\frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\beta^{k+i}=\frac{\alpha^i}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\alpha^k-\frac{\beta^i}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\beta^k=$$
$$=\frac{1}{\sqrt{5}}\left(\alpha^i(1+\alpha)^n-\beta^i(1+\beta)^n\right)=\frac{1}{\sqrt{5}}(\alpha^i\alpha^{2n}-\beta^i\beta^{2n})=\frac{1}{\sqrt{5}}(\alpha^{2n+i}-\beta^{2n+i})=F_{2n+i}.$$

1.58. $\left(\frac{L_n+\sqrt{5}F_n}{2}\right)^m=\frac{L_{mn}+\sqrt{5}F_{mn}}{2}$, $\forall m,n\in\mathbf{N}$ (**Fisk**, 1963).

Proof. $\frac{L_{mn}+\sqrt{5}F_{mn}}{2}=\frac{\alpha^{mn}+\beta^{mn}+\alpha^{mn}-\beta^{mn}}{2}=\alpha^{mn}$, on the other hand $\left(\frac{L_n+\sqrt{5}F_n}{2}\right)^m=\left(\frac{\alpha^n+\beta^n+\alpha^n-\beta^n}{2}\right)^m=(\alpha^n)^m=\alpha^{mn}$, follows the conclusion.

1.59. $L_{2n+1}F_{2n} = F_{4n+1} - 1,\ \forall n \in \mathbf{N}$ (**Hoggatt**, 1967).

Proof. $L_{2n+1}F_{2n} = \frac{1}{\sqrt{5}}(\alpha^{2n+1} + \beta^{2n+1})(\alpha^{2n} - \beta^{2n}) =$

$$= \frac{1}{\sqrt{5}}(\alpha^{4n+1} - \beta^{4n+1} + \alpha^{2n}\beta^{2n+1} - \alpha^{2n+1}\beta^{2n}) = F_{4n+1} + \frac{1}{\sqrt{5}}\left((\alpha\beta)^{2n}\beta - (\alpha\beta)^{2n}\alpha\right) =$$

$$= F_{4n+1} + \frac{1}{\sqrt{5}}(\beta - \alpha) = F_{4n+1} - 1.$$

1.60. $L_{2n+1}F_{2n+2} = F_{4n+3} - 1,\ \forall n \in \mathbf{N}$ (**Hoggatt**, 1967).

Proof. $L_{2n+1}F_{2n+2} = \frac{1}{\sqrt{5}}(\alpha^{2n+1} + \beta^{2n+1})(\alpha^{2n+2} - \beta^{2n+2}) =$

$$= \frac{1}{\sqrt{5}}(\alpha^{4n+3} - \beta^{4n+3} - \alpha^{2n+1}\beta^{2n+2} + \alpha^{2n+2}\beta^{2n+1}) = F_{4n+3} - \frac{1}{\sqrt{5}}\left((\alpha\beta)^{2n+1}\beta - (\alpha\beta)^{2n+1}\alpha\right) =$$

$$= F_{4n+3} + \frac{1}{\sqrt{5}}(-\alpha + \beta) = F_{4n+3} + \frac{1}{\sqrt{5}}\left(-\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right) = F_{4n+3} - 1.$$

1.61. $F_{m+n} = F_m L_n - (-1)^n F_{m-n},\ \forall m, n \in \mathbf{N}, m \geq n$ (**Ruggles**, 1970).

Proof. $F_m L_n - (-1)^n F_{m-n} = \frac{1}{\sqrt{5}}\left((\alpha^m - \beta^m)(\alpha^n + \beta^n) - (\alpha\beta)^n(\alpha^{m-n} - \beta^{m-n})\right) =$

$$= \frac{1}{\sqrt{5}}\left(\alpha^{m+n} + \alpha^m\beta^n - \alpha^n\beta^m - \beta^{m+n} - \alpha^m\beta^n - \alpha^n\beta^m\right) = \frac{1}{\sqrt{5}}(\alpha^{m+n} - \beta^{m+n}) = F_{m+n}.$$

1.62. $L_n L_{n+1} = L_{2n+1} + (-1)^n,\ \forall n \in \mathbf{N}$ (**Hoggatt**, 1965).

Proof. $L_n L_{n+1} = (\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1}) = \alpha^{2n+1} + \beta^{2n+1} + \alpha^n\beta^{n+1} + \alpha^{n+1}\beta =$

$$= L_{2n+1} + (\alpha\beta)^n(\alpha + \beta) = L_{2n+1} + (\alpha\beta)^n = L_{2n+1} + (-1)^n.$$

1.63. $5(F_n^2 + F_{n-2}^2) = 3L_{2n-2} - 4(-1)^n,\ \forall n \in \mathbf{N}^*$.

Proof. $5(F_n^2 + F_{n-2}^2) = 5\left(\frac{1}{5}(\alpha^n - \beta^n)^2 + \frac{1}{5}(\alpha^{n-2} - \beta^{n-2})^2\right) =$

$$= \alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n + \alpha^{2n-4} + \beta^{2n-4} - 2(\alpha\beta)^{n-2} = \alpha^{2n-2}(\alpha^2 + \alpha^{-2}) +$$

$$+ \beta^{2n-2}(\beta^2 + \beta^{-2}) - 4(\alpha\beta)^n = \alpha^{2n-2}(\alpha^2 + \beta^2) + \beta^{2n-2}(\alpha^2 + \beta^2) - 4(-1)^n =$$

$$= (\alpha^2 + \beta^2)(\alpha^{2n-2} + \beta^{2n-2}) - 4(-1)^n = (\alpha^2 + \beta^2)L_{2n-2} - 4(-1)^n =$$

$$= 3L_{2n-2} - 4(-1)^n.$$

1.64. $\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n}$, $\forall n \in \mathbf{N}$.

Proof. $\sum_{k=0}^{n}\binom{n}{k}F_k = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\alpha^k - \beta^k) = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\alpha^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\beta^k =$

$$= \frac{1}{\sqrt{5}}(1+\alpha)^n - \frac{1}{\sqrt{5}}(1+\beta)^n = \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = F_{2n}.$$

1.65. $\sum_{k=0}^{n}\binom{n}{k}L_k = L_{2n}$, $\forall n \in \mathbf{N}$.

Proof. $\sum_{k=0}^{n}\binom{n}{k}L_k = \sum_{k=0}^{n}\binom{n}{k}(\alpha^k + \beta^k) = \sum_{k=0}^{n}\binom{n}{k}\alpha^k + \sum_{k=0}^{n}\binom{n}{k}\beta^k =$

$$= (1+\alpha)^n + (1+\beta)^n = \alpha^{2n} + \beta^{2n} = L_{2n}.$$

1.66. $\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_k = -F_n$, $\forall n \in \mathbf{N}^*$.

Proof. $\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_k = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha^k - \beta^k) =$

$$= \frac{1}{\sqrt{5}}\left(\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^k - \sum_{k=0}^{n}\binom{n}{k}(-\beta)^k\right) = \frac{1}{\sqrt{5}}(1-\alpha)^n - \frac{1}{\sqrt{5}}(1-\beta)^n = \frac{1}{\sqrt{5}}(\beta^n - \alpha^n) =$$

$$= -\frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = -F_n.$$

1.67. $\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_k = L_n$, $\forall n \in \mathbf{N}$.

Proof. $\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha^k + \beta^k) =$

$$= \left(\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^k + \sum_{k=0}^{n}\binom{n}{k}(-\beta)^k\right) = (1-\alpha)^n + (1-\beta)^n = \alpha^n + \beta^n = L_n.$$

1.68. $F_{4n} - 1 = F_{2n+1}L_{2n-1}$, $\forall n \in \mathbf{N}^*$ (**Dudley** şi **Tucker**, 1971).

Proof. $F_{2n+1}L_{2n-1}=\frac{1}{\sqrt{5}}(\alpha^{2n+1}-\beta^{2n+1})(\alpha^{2n-1}+\beta^{2n-1})=\frac{1}{\sqrt{5}}(\alpha^{4n}-\alpha^{2n-1}\beta^{2n-1}+$

$$+\alpha^{2n+1}\beta^{2n-1}-\beta^{4n})=F_{4n}-\left((\alpha\beta)^{2n}\frac{\beta}{\alpha}-(\alpha\beta)^{2n}\frac{\alpha}{\beta}\right)\frac{1}{\sqrt{5}}=F_{4n}-\frac{1}{\sqrt{5}}\left(\frac{\beta}{\alpha}-\frac{\alpha}{\beta}\right)=$$

$$=F_{4n}-\frac{1}{\sqrt{5}}\cdot\frac{(\beta-\alpha)(\alpha+\beta)}{\alpha\beta}=F_{4n}-1.$$

1.69. $F_{4n+1}-1=F_{2n}L_{2n+1},\ \forall n\in\mathbf{N}.$

Proof. $F_{2n}L_{2n+1}=\frac{1}{\sqrt{5}}(\alpha^{2n}-\beta^{2n})(\alpha^{2n+1}+\beta^{2n+1})=\frac{1}{\sqrt{5}}(\alpha^{4n+1}-\alpha^{2n+1}\beta^{2n}+$

$$+\alpha^{2n}\beta^{2n+1}-\beta^{4n+1})=F_{4n+1}-\left((\alpha\beta)^{2n}\alpha-(\alpha\beta)^{2n}\beta\right)\frac{1}{\sqrt{5}}=F_{4n+1}-\frac{1}{\sqrt{5}}(\alpha-\beta)=F_{4n+1}-1.$$

1.70. $F_{4n}+1=F_{2n-1}L_{2n+1},\ \forall n\in\mathbf{N}.$

Proof. $F_{2n-1}L_{2n+1}=\frac{1}{\sqrt{5}}(\alpha^{2n-1}-\beta^{2n-1})(\alpha^{2n+1}+\beta^{2n+1})=\frac{1}{\sqrt{5}}(\alpha^{4n}-\alpha^{2n+1}\beta^{2n-1}+$

$$+\alpha^{2n-1}\beta^{2n+1}-\beta^{4n})=F_{4n}-\left((\alpha\beta)^{2n-1}\alpha^{2}-(\alpha\beta)^{2n-1}\beta^{2}\right)\frac{1}{\sqrt{5}}=F_{4n}+\frac{1}{\sqrt{5}}\left(\alpha^{2}-\beta^{2}\right)=$$

$$=F_{4n}+1.$$

1.71. $F_{4n+1}+1=F_{2n+1}L_{2n},\ \forall n\in\mathbf{N}$.

Proof. $F_{2n+1}L_{2n}=\frac{1}{\sqrt{5}}(\alpha^{2n+1}-\beta^{2n+1})(\alpha^{2n}+\beta^{2n})=\frac{1}{\sqrt{5}}(\alpha^{4n+1}-\alpha^{2n}\beta^{2n+1}+$

$$+\alpha^{2n+1}\beta^{2n}-\beta^{4n+1})=F_{4n+1}+\left((\alpha\beta)^{2n}\alpha-(\alpha\beta)^{2n}\beta\right)\frac{1}{\sqrt{5}}=F_{4n+1}+\frac{1}{\sqrt{5}}(\alpha-\beta)=$$

$$=F_{4n+1}+1.$$

1.72. $F_{4n+2}+1=F_{2n+2}L_{2n},\ \forall n\in\mathbf{N}$.

Proof. $F_{2n+2}L_{2n}=\frac{1}{\sqrt{5}}(\alpha^{2n+2}-\beta^{2n+2})(\alpha^{2n}+\beta^{2n})=\frac{1}{\sqrt{5}}(\alpha^{4n+2}+\alpha^{2n+2}\beta^{2n}-$

$$-\alpha^{2n}\beta^{2n+2}-\beta^{4n+2})=F_{4n+2}+\frac{1}{\sqrt{5}}(\alpha\beta)^{2n}(\alpha^{2}-\beta^{2})=F_{4n+2}+\frac{1}{\sqrt{5}}(\alpha-\beta)(\alpha+\beta)=$$

$$=F_{4n+2}+\frac{1}{\sqrt{5}}(\alpha-\beta)=F_{4n+2}+1.$$

4. ÎN LEGĂTURĂ CU 306-INEQUALITY IN TRIANGLE TRIANGLE MARATHON 301-400 ROMANIAN MATHEMATICAL MAGAZINE 2017

Marin Chirciu[1]

1) In ΔABC

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c}\geq\frac{63r^2-p^2}{4r}.$$

Mehmet Sahin, Ankara, Turkey

Soluție.

Demonstrăm următorul rezultat ajutător:

Lema1.

2) In ΔABC

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c}=\frac{p^2+5r^2+2Rr}{4r}.$$

Demonstrație.

Folosind formulele $m_a^2=\dfrac{2b^2+2c^2-a^2}{4}$ și $h_a=\dfrac{2S}{a}$ obținem:

$$\sum\frac{m_a^2}{h_a}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{a}}=\frac{1}{8S}\sum a\left(2b^2+2c^2-a^2\right)=\frac{1}{8S}\sum a\left(2a^2+2b^2+2c^2-3a^2\right)=$$

$$=\frac{1}{8S}\left(2\sum a\sum a^2-3\sum a^3\right)=\frac{1}{8S}\left[2\cdot 2p\cdot 2\left(p^2-r^2-4Rr\right)-3\cdot 2p\left(p^2-3r^2-6Rr\right)\right]=$$

$$=\frac{p^2+5r^2+2Rr}{4r}.$$

Să trecem la rezolvarea inegalității din enunț:

Folosind **Lema1** inegalitatea se scrie: $\dfrac{p^2+5r^2+2Rr}{4r}\geq\dfrac{63r^2-p^2}{4r}\Leftrightarrow 2p^2\geq 58r^2-2Rr$, care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$. Rămâne să arătăm că:

$2\left(16Rr-5r^2\right)\geq 58r^2-2Rr\Leftrightarrow 34Rr\geq 17r^2\Leftrightarrow R\geq 2r$, evident din inegalitatea lui Euler.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Inegalitatea **1)** poate fi întărită:

3) In ΔABC

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c}\geq\frac{9R}{2}.$$

Soluție.

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Folosind **Lema1** inegalitatea se scrie: $\frac{p^2+5r^2+2Rr}{4r} \geq \frac{9R}{2} \Leftrightarrow p^2 \geq 16Rr-5r^2$, evident din inegalitatea lui Gerretsen.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.
Inegalitatea **3)** este mai tare decât inegalitatea **1)**:

4) In ΔABC

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c} \geq \frac{9R}{2} \geq \frac{63r^2-p^2}{4r}.$$

Soluție.
Vezi inegalitatea **3)** și și inegalitatea lui Gerretsen $p^2 \geq 16Rr-5r^2$.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.
Se poate scrie o inegalitate de sens contrar:

5) In ΔABC

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c} \leq \frac{2R^2+3Rr+4r^2}{2r}.$$

Soluție.
Folosind **Lema1** inegalitatea se scrie: $\frac{p^2+5r^2+2Rr}{4r} \leq \frac{2R^2+3Rr+4r^2}{2r} \Leftrightarrow$

$\Leftrightarrow p^2 \leq 4R^2+4Rr+3r^2$, evident din inegalitatea lui Gerretsen.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.
Se poate scrie dubla inegalitate:

6) In ΔABC

$$\frac{9R}{2} \leq \frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c} \leq \frac{2R^2+3Rr+4r^2}{2r}.$$

Marin Chirciu, Pitești

Soluție.
Vezi inegalitățile **3) și 5).**
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.
În același registru se pot propune:

7) In ΔABC

$$\frac{R}{r}-\frac{r}{4R}+\frac{9}{8} \leq \frac{m_a^2}{h_a^2}+\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2} \leq \frac{R^2}{2r^2}-\frac{3R}{8r}+\frac{7}{4}.$$

Marin Chirciu, Pitești

Soluție.
Demonstrăm următorul rezultat ajutător:

Lema2.

8) In ΔABC

$$\frac{m_a^2}{h_a^2}+\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2}=\frac{p^4+p^2\left(10r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8p^2r^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $h_a=\frac{2S}{a}$ obținem:

$$\sum\frac{m_a^2}{h_a^2}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\left(\frac{2S}{a}\right)^2}=\frac{1}{16S^2}\sum a^2\left(2b^2+2c^2-a^2\right)=\frac{1}{16S^2}\left(4\sum b^2c^2-\sum a^4\right)=$$

$$=\frac{p^4+p^2\left(10r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8p^2r^2}.$$

Să trecem la rezolvarea primei inegalități din **7)** :

Folosind **Lema2** obținem

$$\frac{p^4+p^2\left(10r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8p^2r^2}=\frac{1}{8r^2}\left[p^2+10r^2-8Rr+\frac{r^2\left(4R+r\right)^2}{p^2}\right]\geq$$

$$\geq\frac{1}{8r^2}\left[16Rr-5r^2+10r^2-8Rr+\frac{r^2\left(4R+r\right)^2}{\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}}\right]=\frac{8R^2r+9Rr^2-2r^3}{8Rr^2}=\frac{R}{r}-\frac{r}{4R}+\frac{9}{8},$$

unde ultima inegalitate rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Blundon $p^2\leq\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Pentru cea de-a doua inegalitate scriem:

$$\frac{p^4+p^2\left(10r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8p^2r^2}=\frac{1}{8r^2}\left[p^2+10r^2-8Rr+\frac{r^2\left(4R+r\right)^2}{p^2}\right]\leq$$

$$\leq\frac{1}{8r^2}\left[4R^2+4Rr+3r^2+10r^2-8Rr+\frac{r^2\left(4R+r\right)^2}{\frac{r\left(4R+r\right)^2}{R+r}}\right]=\frac{4R^2-3Rr+14r^2}{8r^2}=\frac{R^2}{2r^2}-\frac{3R}{8r}+\frac{7}{4},$$

unde ultima inegalitate rezultă din inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$ și inegalitatea $p^2\geq\frac{r\left(4R+r\right)^2}{R+r}$, adevărată din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

9) In ΔABC

$$\frac{9}{4}\cdot\frac{R}{r}-\frac{3}{2}\le\frac{m_a^2}{h_bh_c}+\frac{m_b^2}{h_ch_a}+\frac{m_c^2}{h_ah_b}\le\left(\frac{R}{r}\right)^2-\frac{1}{2}\cdot\frac{R}{r}+\frac{1}{2}\cdot\frac{r}{R}-\frac{1}{4}.$$

Marin Chirciu, Pitești

Soluție.

Demonstrăm următorul rezultat ajutător:

Lema3.

10) In ΔABC

$$\frac{m_a^2}{h_bh_c}+\frac{m_b^2}{h_ch_a}+\frac{m_c^2}{h_ah_b}=\frac{p^4-6p^2Rr-r^2(4R+r)^2}{4p^2r^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $h_a=\frac{2S}{a}$ obținem:

$$\sum\frac{m_a^2}{h_bh_c}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{16S^2}\sum bc\left(2b^2+2c^2-a^2\right)=\frac{1}{16S^2}\left[2\sum bc\left(b^2+c^2\right)-abc\sum a\right]=$$

$$=\frac{p^4-6p^2Rr-r^2(4R+r)^2}{4p^2r^2}.$$

Să trecem la rezolvarea primei inegalități din **9)** :

Folosind **Lema3** obținem

$$=\frac{p^4-6p^2Rr-r^2(4R+r)^2}{4p^2r^2}=\frac{1}{4r^2}\left[p^2-6Rr-\frac{r^2(4R+r)^2}{p^2}\right]\ge\frac{1}{4r^2}\left[16Rr-5r^2-6Rr-\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$\frac{10Rr-5r^2-r(R+r)}{4r^2}=\frac{9R-6r}{4r}=\frac{9}{4}\cdot\frac{R}{r}-\frac{3}{2}$$, unde ultima inegalitate rezultă din inegalitatea lui Gerretsen $p^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$.

Pentru cea de-a doua inegalitate scriem:

$$\frac{p^4-6p^2Rr-r^2(4R+r)^2}{4p^2r^2}=\frac{1}{4r^2}\left[p^2-6Rr-\frac{r^2(4R+r)^2}{p^2}\right]\le\frac{1}{4r^2}\left[4R^2+4Rr+3r^2-6Rr-\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{4R^3-2R^2r-Rr^2+2r^3}{4Rr^2}=\left(\frac{R}{r}\right)^2-\frac{1}{2}\cdot\frac{R}{r}+\frac{1}{2}\cdot\frac{r}{R}-\frac{1}{4}$$, unde ultima inegalitate rezultă din inegalitatea lui Gerretsen $p^2\le 4R^2+4Rr+3r^2$ și inegalitatea lui Blundon $p^2\le\frac{R(4R+r)^2}{2(2R-r)}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

11) In ΔABC

$$\frac{m_a^2}{h_bh_c}+\frac{m_b^2}{h_ch_a}+\frac{m_c^2}{h_ah_b}\ge\frac{m_a^2}{h_a^2}+\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2}.$$

Soluție.

Folosind **Lema2** și **Lema3** inegalitatea se scrie:

$$\frac{p^4-6p^2Rr-r^2(4R+r)^2}{4p^2r^2}\ge\frac{p^4+p^2(10r^2-8Rr)+r^2(4R+r)^2}{8p^2r^2}\Leftrightarrow p^4-p^2(4Rr+10r^2)\ge 3r^2(4R+r)^2$$

$\Leftrightarrow p^2(p^2-4Rr-10r^2)\ge 3r^2(4R+r)^2$, care rezultă din inegalitatea lui Gerretsen

$p^2\ge 16Rr-5r^2$ și observația că $p^2-4Rr-10r^2>0$ .Rămâne să arătăm că:

$(16Rr-5r^2)(16Rr-5r^2-4Rr-10r^2)\ge 3r^2(4R+r)^2 \Leftrightarrow 4R^2-9Rr+2r^2\ge 0$

$\Leftrightarrow (R-2r)(4R-r)\ge 0$, evident din inegalitatea lui Euler $R\ge 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

12) In ΔABC

$$\frac{m_a^2}{h_b+h_c}+\frac{m_b^2}{h_c+h_a}+\frac{m_c^2}{h_a+h_b}\ge\frac{9R}{4}.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema4.

13) In ΔABC

$$\frac{m_a^2}{h_b+h_c}+\frac{m_b^2}{h_c+h_a}+\frac{m_c^2}{h_a+h_b}=\frac{p^6+p^4r^2-p^2(36R^2r^2+4Rr^3+r^4)-r^3(4R+r)^3}{4rp^2(p^2+r^2+2Rr)}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $h_a=\frac{2S}{a}$ obținem:

$$\sum\frac{m_a^2}{h_b+h_c}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{b}+\frac{2S}{c}}=\frac{1}{8S}\sum\frac{bc(2b^2+2c^2-a^2)}{b+c}=\frac{1}{8S}\sum\frac{bc(2a^2+2b^2+2c^2-3a^2)}{b+c}=$$

$$=\frac{p^6+p^4r^2-p^2(36R^2r^2+4Rr^3+r^4)-r^3(4R+r)^3}{4rp^2(p^2+r^2+2Rr)}.$$

Să trecem la rezolvarea inegalității propuse.

Folosind **Lema4** inegalitatea se scrie:

$$\frac{p^6+p^4r^2-p^2(36R^2r^2+4Rr^3+r^4)-r^3(4R+r)^3}{4rp^2(p^2+r^2+2Rr)}\ge\frac{9R}{4}\Leftrightarrow$$

$$\Leftrightarrow p^6+p^4r^2-p^2(36R^2r^2+4Rr^3+r^4)-r^3(4R+r)^3\ge 9Rrp^2(p^2+r^2+2Rr)\Leftrightarrow$$

$$\Leftrightarrow p^6+p^4(r^2-9Rr)-p^2(54R^2r^2+13Rr^3+r^4)-r^3(4R+r)^3\ge 0\Leftrightarrow$$

$\Leftrightarrow p^2\left[p^4+p^2\left(r^2-9Rr\right)-\left(54R^2r^2+13Rr^3+r^4\right)\right]\geq r^3\left(4R+r\right)^3$, care rezultă inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și observația că $p^4+p^2\left(r^2-9Rr\right)-\left(54R^2r^2+13Rr^3+r^4\right)>0$.

Rămâne să arătăm că:

$\left(16Rr-5r^2\right)\left[\left(16Rr-5r^2\right)^2+\left(16Rr-5r^2\right)\left(r^2-9Rr\right)-\left(54R^2r^2+13Rr^3+r^4\right)\right]\geq r^3\left(4R+r\right)^3$

$\Leftrightarrow \left(16R-5r\right)\left[\left(16R-5r\right)\left(7R-4r\right)-\left(54R^2+13Rr+r^2\right)\right]\geq\left(4R+r\right)^3 \Leftrightarrow$

$\Leftrightarrow \left(16R-5r\right)\left[58R^2-112Rr+19r^2\right]\geq\left(4R+r\right)^3 \Leftrightarrow 144R^3-355R^2r+142Rr^2-16r^3\geq 0 \Leftrightarrow$

$\Leftrightarrow \left(R-2r\right)\left(144R^2-67Rr+8r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

Înlocuind h_a cu r_a se pot obține noi inegalități.

14) In ΔABC

$$9\left(R-r\right)\leq\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\leq\frac{4R^2-3Rr-r^2}{r}.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema1a.

15) In ΔABC

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}=\frac{p^2-4r^2-7Rr}{r}.$$

Demonstrație.

Folosind formulele $m_a^2=\dfrac{2b^2+2c^2-a^2}{4}$ și $r_a=\dfrac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{r_a}=\frac{\dfrac{2b^2+2c^2-a^2}{4}}{\dfrac{S}{p-a}}=\frac{1}{4S}\sum\left(p-a\right)\left(2b^2+2c^2-a^2\right)=\frac{p^2-4r^2-7Rr}{r}$$

Să trecem la rezolvarea inegalității din enunț:

Folosind **Lema1a** dubla inegalitate se scrie: $9\left(R-r\right)\leq\dfrac{p^2-4r^2-7Rr}{r}\leq\dfrac{4R^2-3Rr-r^2}{r}$, care rezultă din inegalitatea lui Gerretsen $16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

16) In ΔABC

$$\frac{4R}{r}-\frac{r}{R}-\frac{9}{2}\leq\frac{m_a^2}{r_a^2}+\frac{m_b^2}{r_b^2}+\frac{m_c^2}{r_c^2}\leq\frac{4R^2}{r^2}-\frac{15R}{2r}+2.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema2a.

17) In ΔABC

$$\frac{m_a^2}{r_a^2}+\frac{m_b^2}{r_b^2}+\frac{m_c^2}{r_c^2}=\frac{2p^4-3p^2\left(r^2+8Rr\right)+r^2\left(4R+r\right)^2}{2p^2r^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $r_a=\frac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{r_a^2}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\left(\frac{S}{p-a}\right)^2}=\frac{1}{4S^2}\sum\left(p-a\right)^2\left(2b^2+2c^2-a^2\right)=\frac{2p^4-3p^2\left(r^2+8Rr\right)+r^2\left(4R+r\right)^2}{2p^2r^2}$$ Să trecem la rezolvarea dublei inegalități **16)** :

Folosind **Lema2a** dubla inegalitate **16)** se scrie:

$\frac{4R}{r}-\frac{r}{R}-\frac{9}{2}\le\frac{2p^4-3p^2\left(r^2+8Rr\right)+r^2\left(4R+r\right)^2}{2p^2r^2}\le\frac{4R^2}{r^2}-\frac{15R}{2r}+2$, care rezultă din

$\frac{2p^4-3p^2\left(r^2+8Rr\right)+r^2\left(4R+r\right)^2}{2p^2r^2}=\frac{1}{2r^2}\left[2p^2-3\left(r^2+8Rr\right)+\frac{r^2\left(4R+r\right)^2}{p^2}\right]$,

inegalitatea lui Gerretsen $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$ și inegalitatea lui Blundon- Gerretsen $\frac{r\left(4R+r\right)^2}{R+r}\le p^2\le\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

18) In ΔABC

$$3\le\frac{m_a^2}{r_br_c}+\frac{m_b^2}{r_cr_a}+\frac{m_c^2}{r_ar_b}\le\frac{R}{r}+\frac{2r}{R}.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema3a.

19) In ΔABC

$$\frac{m_a^2}{r_br_c}+\frac{m_b^2}{r_cr_a}+\frac{m_c^2}{r_ar_b}=\frac{p^2\left(4r^2+Rr\right)-r^2\left(4R+r\right)^2}{p^2r^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $r_a=\frac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{r_br_c}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{S}{p-b}\cdot\frac{S}{p-c}}=\frac{1}{4S^2}\sum(p-b)(p-c)\left(2b^2+2c^2-a^2\right)=\frac{p^2\left(4r^2+Rr\right)-r^2\left(4R+r\right)^2}{p^2r^2}$$ Să trecem la rezolvarea dublei inegalităţi 1**8)** :

Folosind **Lema3a** dubla inegalitate 1**8)** se scrie: $3\le\frac{p^2\left(4r^2+Rr\right)-r^2\left(4R+r\right)^2}{p^2r^2}\le\frac{R}{r}+\frac{2r}{R}$, care rezultă din

$$\frac{p^2\left(4r^2+Rr\right)-r^2\left(4R+r\right)^2}{p^2r^2}=\frac{1}{r^2}\left[4r^2+Rr-\frac{r^2\left(4R+r\right)^2}{p^2}\right],$$

inegalitatea lui Gerretsen $16Rr-5r^2\le p^2\le4R^2+4Rr+3r^2$ şi inegalitatea Blundon- Gerretsen

$$\frac{r\left(4R+r\right)^2}{R+r}\le p^2\le\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarcă.

20) In ΔABC

$$\frac{m_a^2}{r_a^2}+\frac{m_b^2}{r_b^2}+\frac{m_c^2}{r_c^2}\ge\frac{m_a^2}{r_br_c}+\frac{m_b^2}{r_cr_a}+\frac{m_c^2}{r_ar_b}.$$

Soluţie.

Folosind **Lema2a** şi **Lema3a** inegalitatea se scrie:

$$\frac{2p^4-3p^2\left(r^2+8Rr\right)+r^2\left(4R+r\right)^2}{2p^2r^2}\ge\frac{p^2\left(4r^2+Rr\right)-r^2\left(4R+r\right)^2}{p^2r^2}\Leftrightarrow$$

$$\Leftrightarrow p^2\left(2p^2-11Rr-26r^2\right)+3r^2\left(4R+r\right)^2\ge0.$$

Distingem cazurile:

Cazul 1). Dacă $2p^2-11Rr-26r^2\ge0$, inegalitatea este evidentă.

Cazul 2). Dacă $2p^2-11Rr-26r^2<0$, inegalitatea se rescrie:

$3r^2\left(4R+r\right)^2\ge p^2\left(11Rr+26r^2-2p^2\right)$, care rezultă din inegalitatea lui Gerretsen $16Rr-5r^2\le p^2\le4R^2+4Rr+3r^2$. Rămâne să arătăm că:

$$3r^2\left(4R+r\right)^2\ge\left(4R^2+4Rr+3r^2\right)\left(11Rr+26r^2-2\left(16Rr-5r^2\right)\right)$$

$\Leftrightarrow 4R^3-2R^2r-7Rr^2-10r^3\ge0\Leftrightarrow\left(R-2r\right)\left(4R^2+6Rr+5r^2\right)\ge0$, evident din inegalitatea lui Euler $R\ge2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

21) In ΔABC

$$\frac{3r\left(7R-2r\right)}{4R}\le\frac{m_a^2}{r_b+r_c}+\frac{m_b^2}{r_c+r_a}+\frac{m_c^2}{r_a+r_b}\le\frac{8R^3-2R^2r+7Rr^2+2r^3}{4R^2}.$$

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Soluţie.

Demonstrăm următorul rezultat ajutător:

Lema4a.

22) In ΔABC

$$\frac{m_a^2}{r_b+r_c}+\frac{m_b^2}{r_c+r_a}+\frac{m_c^2}{r_a+r_b}=\frac{p^4+p^2\left(4R^2+10Rr\right)-r\left(4R+r\right)^3}{4Rp^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ și $r_a=\frac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{r_b+r_c}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{S}{p-b}+\frac{S}{p-c}}=\frac{1}{4S}\sum\frac{(p-b)(p-c)\left(2b^2+2c^2-a^2\right)}{a}=\frac{p^4+p^2\left(4R^2+10Rr\right)-r\left(4R+r\right)^3}{4Rp^2}$$

Să trecem la rezolvarea dublei inegalități **21)** :

Folosind **Lema4a** dubla inegalitate **21)** se scrie:

$$\frac{3r(7R-2r)}{4R}\le\frac{p^4+p^2\left(4R^2+10Rr\right)-r\left(4R+r\right)^3}{4Rp^2}\le\frac{8R^3-2R^2r+7Rr^2+2r^3}{4R^2},$$ care rezultă din

$$\frac{p^4+p^2\left(4R^2+10Rr\right)-r\left(4R+r\right)^3}{4Rp^2}=\frac{1}{4R}\left[p^2+4R^2+10Rr-\frac{r\left(4R+r\right)^3}{p^2}\right],$$

inegalitatea lui Gerretsen $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$ și inegalitatea Blundon- Gerretsen

$$\frac{r\left(4R+r\right)^2}{R+r}\le p^2\le\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

23) In ΔABC

$$\frac{9R}{4}-\frac{3r}{2}\le\frac{m_a^2}{h_a+2r_a}+\frac{m_b^2}{h_b+2r_b}+\frac{m_c^2}{h_c+2r_c}\le\frac{10R^2-9Rr+2r^2}{4R}.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema5.

24) In ΔABC

$$\frac{m_a^2}{h_a+2r_a}+\frac{m_b^2}{h_b+2r_b}+\frac{m_c^2}{h_c+2r_c}=\frac{p^2\left(10R-5r\right)-r\left(4R+r\right)^2}{4p^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, $h_a=\frac{2S}{a}$ și $r_a=\frac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{h_a+2r_a}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{a}+\frac{2S}{p-a}}=\frac{1}{8S}\sum\frac{a(p-a)\left(2b^2+2c^2-a^2\right)}{p}=\frac{p^2\left(10R-5r\right)-r\left(4R+r\right)^2}{4p^2}.$$

Să trecem la rezolvarea dublei inegalități **23)** :
Folosind **Lema5** dubla inegalitate **23)** se scrie:

$\frac{9R}{4}-\frac{3r}{2}\le\frac{p^2(10R-5r)-r(4R+r)^2}{4p^2}\le\frac{10R^2-9Rr+2r^2}{4R}$, care rezultă din

$\frac{p^2(10R-5r)-r(4R+r)^2}{4p^2}=\frac{1}{4}\left[10R-5r-\frac{r(4R+r)^2}{p^2}\right]$ și inegalitatea Blundon- Gerretsen

$\frac{r(4R+r)^2}{R+r}\le p^2\le\frac{R(4R+r)^2}{2(2R-r)}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

25) In ΔABC

$$\frac{m_a^2}{h_a+2r}+\frac{m_b^2}{h_b+2r}+\frac{m_c^2}{h_c+2r}\ge\frac{27r}{5}.$$

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Soluție.

Demonstrăm următorul rezultat ajutător:

Lema6.

26) In ΔABC

$$\frac{m_a^2}{h_a+2r}+\frac{m_b^2}{h_b+2r}+\frac{m_c^2}{h_c+2r}=\frac{2p^4+p^2(26Rr+5r^2)-r^2(4R+r)(16R+r)}{4r(4p^2+r^2+8Rr)}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, $h_a=\frac{2S}{a}$ și $r=\frac{S}{p}$ obținem:

$$\sum\frac{m_a^2}{h_a+2r}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{a}+\frac{2S}{p}}=\frac{1}{8S}\sum\frac{a(p-a)(2b^2+2c^2-a^2)}{p+a}=$$

$$=\frac{2p^4+p^2(26Rr+5r^2)-r^2(4R+r)(16R+r)}{4r(4p^2+r^2+8Rr)}.$$

Să trecem la rezolvarea inegalități **25)** :
Folosind **Lema6** inegalitatea **25)** se scrie:

$$\Leftrightarrow\frac{2p^4+p^2(26Rr+5r^2)-r^2(4R+r)(16R+r)}{4r(4p^2+r^2+8Rr)}\ge\frac{27r}{5}\Leftrightarrow$$

$$\Leftrightarrow 10p^4+p^2(130Rr-407r^2)\ge 320R^2r^2+964Rr^3+113r^4\Leftrightarrow$$

$\Leftrightarrow\ p^2(10p^2+130Rr-407r^2)\ge 320R^2r^2+964Rr^3+113r^4$,adevărată din inegalitatea lui Gerretsen $p^2\ge 16Rr-5r^2$ .Rămâne să arătăm că:

$$(16Rr-5r^2)(10(16Rr-5r^2)+130Rr-407r^2)\ge 320R^2r^2+964Rr^3+113r^4\Leftrightarrow$$

$\Leftrightarrow 720R^2 - 1621Rr + 362r^2 \geq 0 \Leftrightarrow (R-2r)(720R-181r) \geq 0$, evident din inegalitatea lui Euler $R \geq 2r$.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

27) In ΔABC

$$27r \leq \frac{m_a^2}{h_a-2r}+\frac{m_b^2}{h_b-2r}+\frac{m_c^2}{h_c-2r} \leq \frac{3R^3+30r^3}{2r^2}.$$

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Soluție.
Demonstrăm următorul rezultat ajutător:
Lema7.

28) In ΔABC

$$\frac{m_a^2}{h_a-2r}+\frac{m_b^2}{h_b-2r}+\frac{m_c^2}{h_c-2r} = \frac{p^2(2R+5r)+r(r+4R)(r-8R)}{4r^2}.$$

Demonstrație.

Folosind formulele $m_a^2 = \dfrac{2b^2+2c^2-a^2}{4}$, $h_a = \dfrac{2S}{a}$ și $r = \dfrac{S}{p}$ obținem:

$$\sum \frac{m_a^2}{h_a-2r} = \frac{\dfrac{2b^2+2c^2-a^2}{4}}{\dfrac{2S}{a}-\dfrac{2S}{p}} = \frac{1}{8S}\sum \frac{pa\left(2b^2+2c^2-a^2\right)}{p-a} = \frac{p^2(2R+5r)+r(r+4R)(r-8R)}{4r^2}$$

Să trecem la rezolvarea dublei inegalități **27)** :
Folosind **Lema7** dubla inegalitate **27)** se scrie:

$$\Leftrightarrow 27r \leq \frac{p^2(2R+5r)+r(r+4R)(r-8R)}{4r^2} \leq \frac{3R^3+30r^3}{2r^2}$$

$$\Leftrightarrow 10p^4 + p^2\left(130Rr - 407r^2\right) \geq 320R^2r^2 + 964Rr^3 + 113r^4 \Leftrightarrow$$

$\Leftrightarrow p^2\left(10p^2 + 130Rr - 407r^2\right) \geq 320R^2r^2 + 964Rr^3 + 113r^4$,adevărată din inegalitatea lui Gerretsen $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

29) In ΔABC

$$\frac{9}{4}\cdot\frac{R}{r}-\frac{3}{2} \leq \frac{m_a^2}{h_ar_a}+\frac{m_b^2}{h_br_b}+\frac{m_c^2}{h_cr_c} \leq \frac{5}{2}\cdot\frac{R}{r}+\frac{1}{2}\cdot\frac{R}{r}-\frac{9}{4}.$$

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Soluție.
Demonstrăm următorul rezultat ajutător:
Lema8.

30) In ΔABC

$$\frac{m_a^2}{h_ar_a}+\frac{m_b^2}{h_br_b}+\frac{m_c^2}{h_cr_c} = \frac{p^2\left(10Rr-5r^2\right)-r^2(4R+r)^2}{4p^2r^2}.$$

Demonstrație.

Folosind formulele $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, $h_a=\frac{2S}{a}$ și $r_a=\frac{S}{p-a}$ obținem:

$$\sum\frac{m_a^2}{h_a r_a}=\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{2S}{a}\cdot\frac{S}{p-a}}=\frac{1}{8S^2}\sum a(p-a)\left(2b^2+2c^2-a^2\right)=\frac{p^2\left(10Rr-5r^2\right)-r^2\left(4R+r\right)^2}{4p^2r^2}.$$ Să

trecem la rezolvarea dublei inegalități **29)** :

Folosind **Lema8** dubla inegalitate **29)** se scrie:

$\frac{9}{4}\cdot\frac{R}{r}-\frac{3}{2}\le\frac{p^2\left(10Rr-5r^2\right)-r^2\left(4R+r\right)^2}{4p^2r^2}\le\frac{5}{2}\cdot\frac{R}{r}+\frac{1}{2}\cdot\frac{R}{r}-\frac{9}{4}$, care rezultă din scrierea

$\frac{p^2\left(10Rr-5r^2\right)-r^2\left(4R+r\right)^2}{4p^2r^2}=\frac{1}{4r^2}\left[10Rr-5r^2-\frac{r^2\left(4R+r\right)^2}{p^2}\right]$ și inegalitatea Blundon-Gerretsen $\frac{r\left(4R+r\right)^2}{R+r}\le p^2\le\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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