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1. METODA POLINOAMELOR RECIPROCE(II)

Prof. Gheorghe Ghiţă, Buzău

Articolul propune un alt tip de inegalităţi care se obţin prin utilizarea pozitivităţii unor polinoame reciproce (polinoame care au coeficienţii egali depărtaţi de extreme egali între ei) ce admit pe 1 ca rădăcină dublă. Fie P(t), $t > 0$ polinomul reciproc ataşat inegalităţii respective.

A1. Dacă $a, b, c > 0$, atunci

$$\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{c+a}{b^2} \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

problema O.VIII.537, R.M.T. , nr. 3-4/2023 de Neculai Stanciu, Buzău

Soluţie.

$$\sum \frac{a+b}{c^2} \geq 2\sum\frac{1}{a} \Leftrightarrow \sum\left(\frac{a}{b^2}+\frac{b}{a^2}-\frac{1}{a}-\frac{1}{b}\right)=\sum\frac{a^3+b^3-ab^2-a^2b}{a^2b^2}=\sum\frac{b\left[\left(\frac{a}{b}\right)^3-\left(\frac{a}{b}\right)^2-\frac{a}{b}+1\right]}{a^2}=\sum\frac{bP\left(\frac{a}{b}\right)}{a^2} \geq 0, \text{ unde}$$

$$P(t)=t^3-t^2-t+1=t^2(t-1)-(t-1)=(t-1)^2(t+1) \geq 0.$$

Egalitatea are loc dacă şi numai dacă $a = b = c$.

Extidere A1: Dacă $a, b, c > 0, n, k \in N$ atunci

$$\frac{a^k+b^k}{c^{n+k}}+\frac{b^k+c^k}{a^{n+k}}+\frac{c^k+a^k}{b^{n+k}} \geq 2\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum\frac{a^k+b^k}{c^{n+k}} \geq 2\sum\frac{1}{a^n} \Leftrightarrow \sum\left(\frac{a^k}{b^{n+k}}+\frac{b^k}{a^{n+k}}-\frac{1}{a^n}-\frac{1}{b^n}\right) \geq 0 \Leftrightarrow$$

$$\sum\frac{a^{n+2k}+b^{n+2k}-a^kb^{n+k}-a^{n+k}b^k}{a^{n+k}b^{n+k}}=\sum\frac{b^k\left[\left(\frac{a}{b}\right)^{n+2k}-\left(\frac{a}{b}\right)^{n+k}-\left(\frac{a}{b}\right)^k+1\right]}{a^{n+k}}=\sum\frac{b^kP\left(\frac{a}{b}\right)}{a^{n+k}} \geq 0, \text{ unde}$$

$$P(t)=t^{n+2k}-t^{n+k}-t^k+1=t^{n+k}(t^k-1)-(t^k-1)=(t^k-1)(t^{n+k}-1)=$$

$$(t-1)^2(t^{k-1}+t^{k-2}+\cdots+t+1)(t^{n+k-1}+t^{n+k-2}+\cdots+t+1) \geq 0.$$

Egalitatea are loc dacă şi numai dacă $a = b = c$.

Observaţia 1.

Pentru $n = k = 1 \Rightarrow$ problema O.VIII.537, R.M.T. , nr. 3-4/2023 de Neculai Stanciu, Buzău:

Observaţia 2.

Pentru $n = 4, k = 1$ în inegalitatea anterioară se obţineinegalitatea:

$$\frac{b+c}{a^5} + \frac{c+a}{b^5} + \frac{a+b}{c^5} \geq \frac{2}{3R^4}$$

problema L:535, Sclipirea Minţii, nr,19/2917 de D.M. Bătineţu-Giurgiu,Bucureşti şi Neculai Stanciu,Buzău

Observaţia 3.

Pentru un triunghi ABC şi $n, k \in N$ se obţine inegalitatea:

$$\frac{a^k + b^k}{c^{n+k}} + \frac{b^k + c^k}{a^{n+k}} + \frac{c^k + a^k}{b^{n+k}} \geq \frac{6}{\left(\sqrt{3}R\right)^n},$$

deoarece $\sum \frac{a^k+b^k}{c^{n+k}} \geq 2\sum \frac{1}{a^n} = 2\sum \frac{1^{n+1}}{a^n} \overset{Radon}{\geq} 2\frac{(\sum 1)^{n+1}}{(\sum a)^n} = \frac{2\cdot 3^{n+1}}{(2p)^n} = \frac{2\cdot 3^{n+1}}{2^n p^n} \overset{Mitrinovic}{\geq} \frac{2\cdot 3^{n+1}}{2^n\left(\frac{3\sqrt{3}R}{2}\right)^n} = \frac{6}{\left(\sqrt{3}R\right)^n}$

A2. Dacă $a, b, c > 0; n \in N$ atunci

$$(a^n + b^n + c^n)\left(\frac{1}{a^2} + \frac{1}{a^2} + \frac{1}{a^2}\right) + 3(a^{n-2} + b^{n-2} + c^{n-2}) \geq 2(a^{n-1} + b^{n-1} + c^{n-1})\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^n \sum \frac{1}{a^2} + 3\sum a^{n-2} \geq 2\sum a^{n-1} \sum \frac{1}{a} \Leftrightarrow$$

$$\sum\left(\frac{a^n}{b^2} + \frac{b^n}{a^2}\right) + \sum a^{n-2} + 3\sum a^{n-2} \geq 2\sum\left(\frac{a^{n-1}}{b} + \frac{b^{n-1}}{a}\right) + 2\sum a^{n-2} \Leftrightarrow$$

$$\sum\left(\frac{a^n}{b^2} + \frac{b^n}{a^2}\right) + 2\sum a^{n-2} \geq 2\sum\left(\frac{a^{n-1}}{b} + \frac{b^{n-1}}{a}\right) \Leftrightarrow$$

$$\sum\left(\frac{a^n}{b^2} + \frac{b^n}{a^2} + a^{n-2} + b^{n-2} - \frac{2a^{n-1}}{b} - \frac{2b^{n-1}}{a}\right) = \sum \frac{a^{n+2} - 2a^{n+1}b + a^n b^2 + a^2 b^n - 2ab^{n+1} + b^{n+2}}{a^2 b^2} =$$

$$\sum \frac{b^n}{a^2}\left(\left(\frac{a}{b}\right)^{n+2} - 2\left(\frac{a}{b}\right)^{n+1} + \left(\frac{a}{b}\right)^n - 2\left(\frac{a}{b}\right)^2 + 1\right) = \sum \frac{b^n}{a^2} P\left(\frac{a}{b}\right) \geq 0, \text{ unde}$$

$$P(t)=t^{n+2}-2t^{n+1}+t^n+t^2-2t+1=t^n(t^2-2t+1)+(t^2-2t+1)=$$
$$(t-1)^2(t^n+1)\geq 0.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

Observaţie.

Pentru $n=0\Rightarrow 3\sum\frac{1}{a^2}\geq\left(\sum\frac{1}{a}\right)^2\Leftrightarrow 3\sum x^2\geq(\sum x)^2\Leftrightarrow\sum x^2\geq\sum xy$;

pentru $n=1\Rightarrow\sum a\sum\frac{1}{a^2}+3\sum\frac{1}{a}\geq 6\sum\frac{1}{a}\Leftrightarrow\sum a\sum\frac{1}{a^2}\geq 3\sum\frac{1}{a}\Leftrightarrow\sum\frac{a+b}{c^2}\geq 2\sum\frac{1}{a}$ (aplicaţia A1);

pentru $n=2\Rightarrow\sum a^2\sum\frac{1}{a^2}+9\geq 2\sum a\sum\frac{1}{a}$;

A3. $\mathrm{a,b,c}>0,\mathrm{n}\geq 0\Rightarrow$

$$(a+b+c)\left(\frac{1}{\mathrm{a}^5}+\frac{1}{\mathrm{b}^5}+\frac{1}{c^5}\right)+(2n-3)\left(\frac{1}{\mathrm{a}^4}+\frac{1}{\mathrm{b}^4}+\frac{1}{c^4}\right)\geq 2n\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a\sum\frac{1}{\mathrm{a}^5}+(2n-3)\sum\frac{1}{\mathrm{a}^4}\geq 2n\sum\frac{1}{a^2b^2}\Leftrightarrow$$

$$\sum\left(\frac{a}{b^5}+\frac{b}{a^5}\right)+\sum\frac{1}{\mathrm{a}^4}+(2n-3)\sum\frac{1}{\mathrm{a}^4}\geq 2n\sum\frac{1}{a^2b^2}\Leftrightarrow$$

$$\sum\left(\frac{a}{b^5}+\frac{b}{a^5}\right)+(2n-2)\sum\frac{1}{\mathrm{a}^4}\geq 2n\sum\frac{1}{a^2b^2}\Leftrightarrow$$

$$\sum\left(\frac{a}{b^5}+\frac{b}{a^5}+\frac{n-1}{\mathrm{a}^4}+\frac{n-1}{\mathrm{b}^4}-\frac{2n}{a^2b^2}\right)=\sum\frac{a^6+(n-1)a^5b-2na^3b^3+(n-1)ab^5+b^6}{a^5b^5}=$$

$$\sum\frac{b}{a^5}\left(\left(\frac{a}{b}\right)^6+(n-1)\left(\frac{a}{b}\right)^5-2n\left(\frac{a}{b}\right)^3+(n-1)\frac{a}{b}+1\right)=\sum\frac{b}{a^5}P\left(\frac{a}{b}\right)\geq 0,\text{ unde}$$

$$P(t)=t^6+(n-1)t^5-2nt^3+(n-1)t+1=$$

$$(t^6-t^5-t+1)+nt(t^4-2t^2+1)=(t-1)(t^5-1)+nt(t^2-1)^2=$$

$$(t-1)^2[t^4+(n+1)t^3+(2n+1)t^2+(n+1)t+1]\geq 0.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A4. $\mathrm{a,b,c}>0,\mathrm{n}\in[0,4]\Rightarrow$

$$(a^3+b^3+c^3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+(2n-2)(ab+bc+ca)\geq(2n+1)(a^2+b^2+c^2)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^3 \sum \frac{1}{a} + (2n-2)\sum ab \geq (2n+1)\sum a^2 \Leftrightarrow$$

$$\sum \left(\frac{a^3}{b} + \frac{b^3}{a}\right) + \sum a^2 + (2n-2)\sum ab \geq (2n+1)\sum a^2 \Leftrightarrow$$

$$\sum \left(\frac{a^3}{b} + \frac{b^3}{a}\right) + (2n-2)\sum ab \geq 2n\sum a^2 \Leftrightarrow$$

$$\sum \left(\frac{a^3}{b} + \frac{b^3}{a} + (2n-2)ab - n(a^2+b^2)\right) = \sum \frac{a^4 - na^3b + (2n-2)a^2b^2 - nab^3 + b^4}{ab} =$$

$$\sum \frac{b^4}{ab}\left[\left(\frac{a}{b}\right)^4 - n\left(\frac{a}{b}\right)^3 + (2n-2)\left(\frac{a}{b}\right)^2 - n\frac{a}{b} + 1\right] = \sum \frac{b^3}{a} P\left(\frac{a}{b}\right) \geq 0 \text{ ,unde}$$

$$P(t) = t^4 - nt^3 + (2n-2)t^2 - nt + 1 = (t^4 - 2t^2 + 1) - nt(t^2 - 2t + 1) =$$

$$(t-1)^2[t^2 + (2-n)t + 1] \geq 0, \forall \text{n} \in [0,4].$$

Egalitatea are loc dacă şi numai dacă $a = b = c$.

A5. $\text{a, b, c} > 0; \text{n} \geq -2 \Rightarrow$

$$a^5 + b^5 + c^5 + n[ab(a^3+b^3) + bc(b^3+c^3) + ca(c^3+a^3)] \geq (n+1)[a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a)]$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^5 + n\sum ab(a^3+b^3) \geq (n+1)\sum a^2b^2(a+b) \Leftrightarrow$$

$$\sum[a^5 + na^4b - (n+1)a^3b^2 - (n+1)a^2b^3 + nab^4 + b^5] =$$

$$\sum b^5\left[\left(\frac{a}{b}\right)^5 + n\left(\frac{a}{b}\right)^4 - (n+1)\left(\frac{a}{b}\right)^3 - (n+1)\left(\frac{a}{b}\right)^2 + n\frac{a}{b} + 1\right] = \sum b^5 P\left(\frac{a}{b}\right) \geq 0, \text{ unde}$$

$$P(t) = t^5 + nt^4 - (n+1)t^3 - (n+1)t^2 + nt + 1 =$$

$$(t^5 - t^3 - t^2 + 1) + nt(t^3 - t^2 - t + 1) = (t^3-1)(t^2-1) + nt(t^2-1)(t-1) =$$

$$(t-1)^2[t^3 + (n+2)t^2 + (n+2)t + 1] \geq 0.$$

Egalitatea are loc dacă şi numai dacă $a = b = c$.

A6. $\text{a, b, c} > 0; \text{n} \geq -1 \Rightarrow$

$$(a^4+b^4+c^4)\left(\frac{1}{\text{a}} + \frac{1}{\text{b}} + \frac{1}{c}\right) + (2n-1)(a^3+b^3+c^3) \geq (n+1)[ab(a+b) + bc(b+c) + ca(c+a)]$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^4 \sum \frac{1}{a} + (2n-1)\sum a^3 \geq (n+1)\sum ab(a+b) \Leftrightarrow$$

$$\sum\left(\frac{a^4}{b}+\frac{b^4}{a}\right)+\sum a^3+(2n-1)\sum a^3 \geq (n+1)\sum ab(a+b) \Leftrightarrow$$

$$\sum\left[\frac{a^4}{b}+na^3-(n+1)a^2b-(n+1)ab^2+nb^3+\frac{b^4}{a}\right]=$$

$$\sum\frac{a^5+na^4b-(n+1)a^3b^2-(n+1)a^2b^3+nab^4+b^5}{ab}=\sum\frac{b^5}{ab}\left[\left(\frac{a}{b}\right)^5+n\left(\frac{a}{b}\right)^4-(n+1)\left(\frac{a}{b}\right)^3-(n+1)\left(\frac{a}{b}\right)^2+n\frac{a}{b}+1\right]=\sum\frac{b^4}{a}P\left(\frac{a}{b}\right)\geq 0, \text{ unde}$$

$$P(t)=t^5+nt^4-(n+1)t^3-(n+1)t^2+nt+1=(t-1)^2[t^3+(n+2)t^2+(n+2)t+1]\geq 0.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A7. $a,b,c>0; n\geq -2 \Rightarrow$

$$(a^3+b^3+c^3)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+n(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq(3n+3)(a+b+c)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^3\sum\frac{1}{a^2}+n\sum a^2\sum\frac{1}{a}\geq(3n+3)\sum a \Leftrightarrow$$

$$\sum\left(\frac{a^3}{b^2}+\frac{b^3}{a^2}\right)+\sum a+n\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right)+n\sum a\geq(3n+3)\sum a \Leftrightarrow$$

$$\sum\left(\frac{a^3}{b^2}+\frac{b^3}{a^2}\right)+n\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right)\geq(2n+2)\sum a \Leftrightarrow$$

$$\sum\left[\frac{a^3}{b^2}+\frac{b^3}{a^2}+n\frac{a^2}{b}+n\frac{b^2}{a}-(n+1)a-(n+1)b\right]=$$

$$\sum\frac{a^5+na^4b-(n+1)a^3b^2-(n+1)a^2b^3+nab^4+b^5}{a^2b^2}=$$

$$\sum\frac{b^5}{a^2b^2}\left[\left(\frac{a}{b}\right)^5+n\left(\frac{a}{b}\right)^4-(n+1)\left(\frac{a}{b}\right)^3-(n+1)\left(\frac{a}{b}\right)^2+n\frac{a}{b}+1\right]=\sum\frac{b^3}{a^2}P\left(\frac{a}{b}\right)\geq 0, \text{ unde}$$

$$P(t)=t^5+nt^4-(n+1)t^3-(n+1)t^2+nt+1=(t-1)^2[t^3+(n+2)t^2+(n+2)t+1]\geq 0$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A8. $a,b,c>0, n\geq -3 \Rightarrow$

$$(a^5+b^5+c^5)\left(\frac{1}{\mathrm{a}^2}+\frac{1}{\mathrm{b}^2}+\frac{1}{c^2}\right)+(2n-1)(a^3+b^3+c^3)$$
$$\geq (n+1)[ab(a+b)+bc(b+c)+ca(c+a)]$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^5\sum\frac{1}{\mathrm{a}^2}+(2n-1)\sum a^3\geq (n+1)\sum ab(a+b)\Leftrightarrow$$

$$\sum\left(\frac{a^5}{b^2}+\frac{b^5}{a^2}\right)+\sum a^3+(2n-1)\sum a^3\geq (n+1)\sum ab(a+b)\Leftrightarrow$$

$$\sum\left(\frac{a^5}{b^2}+\frac{b^5}{a^2}\right)++2n\sum a^3\geq (n+1)\sum ab(a+b)\Leftrightarrow$$

$$\sum\left[\frac{a^5}{b^2}+\frac{b^5}{a^2}+n\,a^3+nb^3-(n+1)a^2b-(n+1)ab^2\right]=$$

$$\sum\frac{b^7}{a^2b^2}\left[\left(\frac{a}{b}\right)^7+n\left(\frac{a}{b}\right)^5-(n+1)\left(\frac{a}{b}\right)^4-(n+1)\left(\frac{a}{b}\right)^3+n\left(\frac{a}{b}\right)^2+1\right]=\sum\frac{b^5}{a^2}P\left(\frac{a}{b}\right)\geq 0,\text{ unde}$$

$$P(t)=t^7+nt^5-(n+1)t^4-(n+1)t^3+nt^2+1=$$

$$(t-1)^2(t^5+2t^4+(n+3)t^3+(n+3)t^2+2t+1)\geq 0,\forall n\geq -3.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A9. $\mathrm{a,b,c}>0,\mathrm{n}\geq -3\Rightarrow$

$$(a^4+b^4+c^4)\left(\frac{1}{\mathrm{a}^3}+\frac{1}{\mathrm{b}^3}+\frac{1}{c^3}\right)+n(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq (3n+3)(a+b+c)$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^4\sum\frac{1}{\mathrm{a}^3}+n\sum a^2\sum\frac{1}{a}\geq (3n+3)\sum a\Leftrightarrow$$

$$\sum\left(\frac{a^4}{b^3}+\frac{b^4}{a^3}\right)+\sum a+n\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right)+n\sum a\geq (3n+3)\sum a\Leftrightarrow$$

$$\sum\left(\frac{a^4}{b^3}+\frac{b^4}{a^3}\right)+n\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right)\geq (2n+2)\sum a\Leftrightarrow$$

$$\sum\left[\frac{a^4}{b^3}+\frac{b^4}{a^3}+n\frac{a^2}{b}+n\frac{b^2}{a}-(n+1)a-(n+1)b\right]=$$

$$\sum\frac{a^7+na^5b^2-(n+1)a^4b^3-(n+1)a^3b^4+na^2b^5+b^7}{a^3b^3}=\sum\frac{b^7}{a^3b^3}\left[\left(\frac{a}{b}\right)^7+n\left(\frac{a}{b}\right)^5-(n+1)\left(\frac{a}{b}\right)^4-\right.$$

$$\left.(n+1)\left(\frac{a}{b}\right)^3+n\left(\frac{a}{b}\right)^2+1\right]=\sum\frac{b^4}{a^3}P\left(\frac{a}{b}\right)\geq 0;\text{ unde}$$

$$P(t)=t^7+nt^5-(n+1)t^4-(n+1)t^3+nt^2+1=$$

$$(t-1)^2(t^5+2t^4+(n+3)t^3+(n+3)t^2+2t+1)\geq 0, \forall n\geq -3.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A10. $\mathrm{a,b,c}>0, \propto\leq 2, \mathrm{n}\in \mathrm{N}\Rightarrow$

$$(a^{n+4}+b^{n+4}+c^{n+4})\left(\frac{1}{\mathrm{a}^2}+\frac{1}{\mathrm{b}^2}+\frac{1}{c^2}\right)+\propto[ab(a^n+b^n)+bc(b^n+c^n)+ca(c^n+a^n)]$$
$$\geq(2\propto+3)(a^{n+2}+b^{n+2}+c^{n+2})$$

Gheorghe Ghiţă, Buzău

Soluţie.

$$\sum a^{n+4}\sum\frac{1}{\mathrm{a}^2}+\propto\sum ab(a^n+b^n)\geq(2\propto+3)\sum a^{n+2}\Leftrightarrow$$

$$\sum\left(\frac{a^{n+4}}{b^2}+\frac{b^{n+4}}{a^2}\right)+\sum a^{n+2}+\propto\sum ab(a^n+b^n)\geq(2\propto+3)\sum a^{n+2}\Leftrightarrow$$

$$\sum\left(\frac{a^{n+4}}{b^2}+\frac{b^{n+4}}{a^2}\right)+\propto\sum ab(a^n+b^n)\geq(2\propto+2)\sum a^{n+2}\Leftrightarrow$$

$$\sum\left[\frac{a^{n+4}}{b^2}+\frac{b^{n+4}}{a^2}+\propto ab(a^n+b^n)-(\propto+1)(a^{n+2}+b^{n+2})\right]=$$

$$\sum\frac{a^{n+6}+b^{n+6}+\propto a^{n+3}b^3+\propto a^3b^{n+3}-(\propto+1)a^{n+4}b^2-(\propto+1)a^2b^{n+4}}{a^2b^2}=$$

$$\sum\frac{b^{n+6}}{a^2b^2}\left[\left(\frac{a}{b}\right)^{n+6}-(\propto+1)\left(\frac{a}{b}\right)^{n+4}+\propto\left(\frac{a}{b}\right)^{n+3}+\propto\left(\frac{a}{b}\right)^3-(\propto+1)\left(\frac{a}{b}\right)^2+1\right]=\sum\frac{b^{n+4}}{a^2}P\left(\frac{a}{b}\right)\geq 0, \text{ unde}$$

$$P(t)=t^{n+6}-(\propto+1)t^{n+4}+\propto t^{n+3}+\propto t^3-(\propto+1)t^2+1=$$

$$(t-1)^2\big(t^{n+4}+2t^{n+3}+2t+1+(2-\propto)(t^{n+2}+t^{n+1}+\cdots+t+1)\big)\geq 0.$$

Egalitatea are loc dacă şi numai dacă $a=b=c$.

Art 7000

Math Journal
-4-

Marin Chirciu [1]

Mathematical Journal prezintă o selecție de probleme recente din diverse publicații de specialitate .

Problema255.

If $x, y, z > 0$, $x + y + z = 3$ then

$$\sum \sqrt{\frac{x}{2(x+1)}} \le \frac{3}{2}.$$

Pure Inequalities, Problem(135) 11/24, Konstantinos Geronikolas, Greece

Soluție.

Se aplică inegalitatea lui Jensen pentru funcția concavă $f(x) = \sqrt{\frac{x}{2(x+1)}}, x > 0$ obținem:

$$LHS = f(x) + f(y) + f(z) \le 3f\left(\frac{x+y+z}{3}\right) = 3f(1) = 3\sqrt{\frac{1}{2(1+1)}} = \frac{3}{2} = RHS.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Generalizare.

If $x_1, x_2, ..., x_n > 0$, $x_1 + x_2 + ... + x_n = n$ then

$$\sum \sqrt{\frac{x_1}{2(x_1+1)}} \le \frac{n}{2}.$$

Remarca.

In ΔABC

[1] *Profesor, Colegiul Național „Zinca Golescu"Pitești*

$$1)\sum \frac{1}{\sqrt{3r+r_a}} \le \sqrt{\frac{3}{2r}}.$$

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\sum \sqrt{\frac{x}{2(x+1)}} \le \frac{3}{2}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum \sqrt{\frac{\frac{3r}{r_a}}{2\left(\frac{3r}{r_a}+1\right)}} \le \frac{3}{2} \Leftrightarrow \sum \sqrt{\frac{3r}{2(3r+r_a)}} \le \frac{3}{2} \Leftrightarrow \sum \frac{1}{\sqrt{3r+r_a}} \le \sqrt{\frac{3}{2r}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2)\sum \frac{1}{\sqrt{3r+h_a}} \le \sqrt{\frac{3}{2r}}.$$

Dezvoltări,Marin Chirciu

Problema256.

In ΔABC

$$\sum \sqrt{\frac{r_a}{h_a}} \le \frac{3R}{2r}.$$

Mathematical Inequalities 11/2024, George Apostolopoulos, Greece

Soluție.

Lema.

In ΔABC

$$\sum \frac{r_a}{h_a} = \frac{2R-r}{r}.$$

$$LHS = \sum \sqrt{\frac{r_a}{h_a}} \overset{CBS}{\leq} \sqrt{3\sum \frac{r_a}{h_a}} = \sqrt{3 \cdot \frac{2R-r}{r}} \overset{Euler}{\leq} \frac{3R}{2r} = RHS,$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\left(\frac{2r}{R}\right)^{\frac{1}{6}} \leq \sum \sqrt{\frac{h_a}{r_a}} \leq 3\left(\frac{R}{2r}\right).$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum \frac{h_a}{r_a} = \frac{p^2+r^2-8Rr}{2Rr}.$$

Inegalitatea din dreapta.

$$\sum \sqrt{\frac{h_a}{r_a}} \overset{CBS}{\leq} \sqrt{3\sum \frac{h_a}{r_a}} = \sqrt{3 \cdot \frac{p^2+r^2-8Rr}{4Rr}} \overset{Gerretsen}{\leq} \sqrt{\frac{6\left(R^2-Rr+r^2\right)}{4Rr}} \overset{Euler}{\leq} \frac{3R}{2r}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum \sqrt{\frac{h_a}{r_a}} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \sqrt{\frac{h_a}{r_a}}} = 3\sqrt[6]{\prod \frac{h_a}{r_a}} = 3\sqrt[6]{\frac{2r}{R}} = 3\left(\frac{2r}{R}\right)^{\frac{1}{6}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ 6 \leq \sum \left(\sqrt{\frac{h_a}{r_a}} + \sqrt{\frac{r_a}{h_a}}\right) \leq \frac{3R}{r}.$$

Soluție.

Lema1.

In ΔABC

$$3\left(\frac{2r}{R}\right)^{\frac{1}{6}} \le \sum\sqrt{\frac{h_a}{r_a}} \le 3\left(\frac{R}{2r}\right).$$

$$\sum\frac{h_a}{r_a} = \frac{p^2+r^2-8Rr}{2Rr}.$$

Soluție.

Inegalitatea din dreapta.

$$\sum\sqrt{\frac{h_a}{r_a}} \overset{CBS}{\le} \sqrt{3\sum\frac{h_a}{r_a}} = \sqrt{3\cdot\frac{p^2+r^2-8Rr}{4Rr}} \overset{Gerretsen}{\le} \sqrt{\frac{6\left(R^2-Rr+r^2\right)}{4Rr}} \overset{Euler}{\le} \frac{3R}{2r},$$

Inegalitatea din stânga.

$$\sum\sqrt{\frac{h_a}{r_a}} \overset{AM-GM}{\ge} 3\sqrt[3]{\prod\sqrt{\frac{h_a}{r_a}}} = 3\sqrt[6]{\prod\frac{h_a}{r_a}} = 3\sqrt[6]{\frac{2r}{R}} = 3\left(\frac{2r}{R}\right)^{\frac{1}{6}}.$$

Lema2.

In ΔABC

$$3\left(\frac{R}{2r}\right)^{\frac{1}{6}} \le \sum\sqrt{\frac{r_a}{h_a}} \le \frac{3R}{2r}.$$

$$\sum\frac{r_a}{h_a} = \frac{2R-r}{r}.$$

Inegalitatea din dreapta.

$$\sum\sqrt{\frac{r_a}{h_a}} \overset{CBS}{\le} \sqrt{3\sum\frac{r_a}{h_a}} = \sqrt{3\cdot\frac{2R-r}{r}} \overset{Euler}{\le} \frac{3R}{2r},$$

Inegalitatea din stânga.

$$\sum\sqrt{\frac{r_a}{h_a}} \overset{CBS}{\ge} 3\sqrt[3]{\prod\sqrt{\frac{r_a}{h_a}}} = 3\sqrt[6]{\prod\frac{r_a}{h_a}} = 3\sqrt[6]{\frac{2r}{R}} = 3\left(\frac{2r}{R}\right)^{\frac{1}{6}}.$$

Adunând dublele inegalități din **Lemele** de mai sus obținem concluzia.

$$3)\sum \frac{h_a}{r_a} \le \sum \frac{r_a}{h_a}.$$

$$4)\ 3\sqrt{2} \le \sum \sqrt{\frac{h_a}{r_a}+\frac{r_a}{h_a}} \le \frac{3\sqrt{2}R}{2r}.$$

Dezvotări, Marin Chirciu

$$\sum\left(\frac{h_a}{r_a}+\frac{r_a}{h_a}\right)=\frac{p^2+4R^2-10Rr+r^2}{2Rr}.$$

Soluție.

Inegalitatea din dreapta.

$$\sum \sqrt{\frac{h_a}{r_a}+\frac{r_a}{h_a}} \overset{CBS}{\le} \sqrt{3\sum\left(\frac{h_a}{r_a}+\frac{r_a}{h_a}\right)} \overset{(1)}{\le} \sqrt{3\cdot\frac{3R^2}{2r^2}} \overset{Euler}{\le} \frac{3\sqrt{2}R}{2r},$$

$$\sum\left(\frac{h_a}{r_a}+\frac{r_a}{h_a}\right)=\frac{p^2+4R^2-10Rr+r^2}{2Rr} \overset{Gerretsen}{\le} \frac{4R^2+4Rr+3r^2+4R^2-10Rr+r^2}{2Rr}=$$

$$=\frac{8R^2-6Rr+4r^2}{2Rr}=\frac{4R^2-3Rr+2r^2}{Rr} \overset{Euler}{\le} \frac{\frac{3R^3}{2r}}{Rr}=\frac{3R^2}{2r^2}.$$

Inegalitatea din stânga.

$$\sum \sqrt{\frac{h_a}{r_a}+\frac{r_a}{h_a}} \overset{AM-GM}{\ge} 3\sqrt{2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema257.

If $x, y, z > 0$, $x+y+z=1$ then find min of

$$P=\sum \frac{x^6}{y^3+z^3}.$$

RMM 11/2024,Socrate Romanidis, Greece

Soluție.

$$P=\sum\frac{x^6}{y^3+z^3}\overset{CS}{\geq}\frac{\left(\sum x^3\right)^2}{\sum\left(y^3+z^3\right)}=\frac{\left(\sum x^3\right)^2}{2\sum x^3}=\frac{1}{2}\sum x^3\overset{Holder}{\geq}\frac{1}{2}\cdot\frac{\left(\sum x\right)^3}{9}=\frac{1}{2}\cdot\frac{1^3}{9}=\frac{1}{18}.$$

Rezultă că $\min P=\frac{1}{18}$ pentru $x=y=z=\frac{1}{3}$.

Remarca.

Let $\lambda\geq 0$ fixed. If $x,y,z>0$, $x+y+z=1$ then find min of

$$P=\sum\frac{x^6}{y^3+\lambda z^3}.$$

Marin Chirciu

Soluție.

$$P=\sum\frac{x^6}{y^3+\lambda z^3}\overset{CS}{\geq}\frac{\left(\sum x^3\right)^2}{\sum\left(y^3+\lambda z^3\right)}=\frac{\left(\sum x^3\right)^2}{(\lambda+1)\sum x^3}=\frac{1}{\lambda+1}\sum x^3\overset{Holder}{\geq}\frac{1}{\lambda+1}\cdot\frac{\left(\sum x\right)^3}{9}=$$

$$=\frac{1}{\lambda+1}\cdot\frac{1^3}{9}=\frac{1}{9(\lambda+1)}.$$

Rezultă că $\min P=\frac{1}{9(\lambda+1)}$ pentru $x=y=z=\frac{1}{3}$.

Problema258.

If $a,b,c>0$, $a^2+b^2+c^2+abc=4$ then

$$\sum\frac{1}{a^2}\geq 2\sum\frac{a}{bc};$$

RMM 11/2024,Nguyen Hung Cuong, Vietnam

Soluție.

Cu substituția $(x,y,z)=\left(\frac{bc}{a},\frac{ca}{b},\frac{ab}{c}\right)$ avem $\left(a^2,b^2,c^2\right)=(yz,zx,xy)$.

Problema se reformulează:

If $x,y,z>0$, $yz+zx+xy+xyz=4$ then

$$\sum x \geq 2\sum xy .$$

Demonstrație.

$$yz + zx + xy + xyz = 4 \Leftrightarrow \sum \frac{1}{x+1} = 2 \Leftrightarrow \sum \frac{x}{x+1} = 1 .$$

$$1 = \sum \frac{x}{x+1} = \sum \frac{x^2}{x^2+x} \overset{CS}{\geq} \frac{\left(\sum x\right)^2}{\sum\left(x^2+x\right)} = \frac{\sum x^2 + 2\sum xy}{\sum x^2 + \sum x} \Rightarrow$$

$$\Rightarrow 1 \geq \frac{\sum x^2 + 2\sum xy}{\sum x^2 + \sum x} \Leftrightarrow \sum x^2 + \sum x \geq \sum x^2 + 2\sum xy \Leftrightarrow \sum x \geq 2\sum xy .$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1 \Leftrightarrow a = b = c = 1$.

Remarca.

If $a,b,c > 0$, $a^2 + b^2 + c^2 + abc = 4$ and $n \in \mathbf{N}$ then

$$\sum \frac{1}{a^{2n}} \geq 3\left(\frac{2}{3}\sum \frac{a}{bc}\right)^n ;$$

Marin Chirciu

Soluție.

Pentru $n = 0$ se obține egalitatea $3 = 3$.

Pentru $n = 1$ se obține **Lema**.

Pentru $n \geq 2$ se folosește inegalitatea lui Holder.

Lema.

If $a,b,c > 0$, $a^2 + b^2 + c^2 + abc = 4$ then

$$\sum \frac{1}{a^2} \geq 2\sum \frac{a}{bc} ;$$

$$LHS = \sum \frac{1}{a^{2n}} \overset{Holder}{\geq} \frac{\left(\sum \frac{1}{a^2}\right)^n}{3^{n-1}} \overset{Lema}{\geq} \frac{\left(2\sum \frac{a}{bc}\right)^n}{3^{n-1}} = 3\left(\frac{2}{3}\sum \frac{a}{bc}\right)^n = RHS .$$

Egalitatea are loc dacă și numai dacă $a = b = c = 1$.

Problema259.

If $a,b,c>0$, $abc=1$ then

1) $\sum\frac{1+ab}{1+a}\geq 3$;

2) $\sum\frac{1+ab}{1+c}\geq 3$;

3) $\sum\frac{1+ab}{1+\sqrt{c}}\geq 3$.

Mathematical Inequalities 11/2024, Imad Zak, Lebanon

Soluție.

1) $\sum\frac{1+ab}{1+a}\geq 3$;

$$LHS=\sum\frac{1+ab}{1+a}=\sum\frac{1+ab}{1+\frac{1}{bc}}=\sum\frac{bc(1+ab)}{1+bc}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{bc(1+ab)}{1+bc}}=3\sqrt[3]{(abc)^2}=3=RHS.$$

2) $\sum\frac{1+ab}{1+c}\geq 3$;

$$\sum\frac{1+ab}{1+c}=\sum\frac{1+ab}{1+\frac{1}{ab}}=\sum\frac{ab(1+ab)}{1+ab}=\sum ab\overset{AM-GM}{\geq}3\sqrt[3]{(abc)^2}=3=RHS.$$

3) $\sum\frac{1+ab}{1+\sqrt{c}}\geq 3$.

$$\sum\frac{1+ab}{1+\sqrt{c}}\overset{CS}{\geq}\sum\frac{1+\frac{1}{c}}{\sqrt{2(1+c)}}=\sum\frac{1+c}{c\sqrt{2(1+c)}}=\sum\frac{\sqrt{1+c}}{c\sqrt{2}}\overset{AM-GM}{\geq}\sum\frac{\sqrt{2\sqrt{c}}}{c\sqrt{2}}=\sum\frac{\sqrt{\sqrt{c}}}{c}=\sum\frac{\sqrt[4]{c}}{c}=$$

$$=\sum\frac{1}{\sqrt[4]{c^3}}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{\sqrt[4]{(abc)^3}}}=3=RHS.$$

Remarca.

If $a,b,c,d>0$, $abcd=1$ then

$$\sum\frac{1+abc}{1+a}\geq 4.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{1+abc}{1+a}=\sum\frac{1+abc}{1+\frac{1}{bcd}}=\sum\frac{bcd(1+abc)}{1+bcd}\overset{AM-GM}{\geq}4\sqrt[4]{\prod\frac{bcd(1+abc)}{1+bcd}}=$$

$$=4\sqrt[4]{(abcd)^3}=4=RHS.$$

Problema260.

If $x,y,z>0$ then

$$\sum\frac{x^4+1}{y^3+y^2+y}\geq 2.$$

Mathematical Inequalities 11/2024

Soluție.

Lema.

If $x,y>0$ then

$$\frac{x^4+1}{x^3+x^2+x}\geq\frac{2}{3}.$$

Demonstrație.

$$\frac{x^4+1}{x^3+x^2+x}\geq\frac{2}{3}\Leftrightarrow 3x^4-2x^3-2x^2-2x+3\geq 0\Leftrightarrow (x-1)^2\left(3x^2+4x+3\right)\geq 0.$$

$$\sum\frac{x^4+1}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^4+1}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{x^4+1}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod\frac{2}{3}}=2=RHS.$$

Remarca.

If $x,y,z>0$ then

$$1)\ \sum\frac{2x^5+3}{y^3+y^2+y}\geq 5.$$

Soluție.

If $x,y>0$ then

$$\frac{2x^5+3}{x^3+x^2+x}\geq\frac{5}{3}.$$

$$\frac{2x^5+3}{x^3+x^2+x}\geq\frac{5}{3}\Leftrightarrow 6x^5-5x^3-5x^2-5x+9\geq 0\Leftrightarrow (x-1)^2\left(6x^3+12x^2+13x+9\right)\geq 0.$$

$$\sum\frac{2x^5+3}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{2x^5+3}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{2x^5+3}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod\frac{5}{3}}=5=RHS.$$

$$2)\sum\frac{x^6+2}{y^3+y^2+y}\geq 3.$$

Soluție.

$$\frac{x^6+2}{x^3+x^2+x}\geq 1.$$

$$\frac{x^6+2}{x^3+x^2+x}\geq 1\Leftrightarrow x^6-x^3-x^2-x+2\geq 0\Leftrightarrow (x-1)^2\left(x^4+2x^3+3x^2+3x+2\right)\geq 0.$$

$$\sum\frac{x^6+2}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^6+2}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{x^6+2}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod 1}=3=RHS.$$

$$3)\sum\frac{2x^7+5}{y^3+y^2+y}\geq 7.$$

Soluție.

$$\frac{2x^7+5}{x^3+x^2+x}\geq\frac{7}{3}.$$

$$\frac{2x^7+5}{x^3+x^2+x}\geq\frac{7}{3}\Leftrightarrow 6x^7-7x^3-7x^2-7x+15\geq 0\Leftrightarrow$$
$$(x-1)^2\left(6x^5+12x^4+18x^3+24x^2+23x+15\right)\geq 0.$$

$$\sum\frac{2x^7+5}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{2x^7+5}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{2x^7+5}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod\frac{7}{3}}=7=RHS.$$

$$4)\sum\frac{x^8+3}{y^3+y^2+y}\geq 4.$$

Soluție.

$$\frac{x^8+3}{x^3+x^2+x}\geq\frac{4}{3}.$$

$$\frac{x^8+3}{x^3+x^2+x}\geq\frac{4}{3}\Leftrightarrow 3x^8-4x^3-4x^2-4x+9\geq 0\Leftrightarrow$$

$$\Leftrightarrow (x-1)^2\left(3x^6+6x^5+9x^4+12x^3+15x^2+14x+9\right)\geq 0.$$

$$\sum\frac{x^8+3}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^8+3}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{x^8+3}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod\frac{4}{3}}=4=RHS.$$

Remarca.

$$5)\sum\frac{x^{2n}+n-1}{y^3+y^2+y}\geq n,\, n\in\mathbf{N}, n\geq 2.$$

Dezvoltări, Marin Chirciu

Soluție.

$$\frac{x^{2n}+n-1}{y^3+y^2+y}\geq\frac{n}{3}.$$

$$\frac{x^{2n}+n-1}{y^3+y^2+y}\geq\frac{n}{3}\Leftrightarrow 3x^{2n}-nx^3-nx^2-nx+3n-3\geq 0\Leftrightarrow$$

$$\Leftrightarrow (x-1)^2\left(3x^{2n-2}+6x^{2n-3}+9x^{2n-4}+\ldots+(6n-12)x^3+(6n-9)x^2+(5n-6)x+3n-3\right)\geq 0.$$

$$\sum\frac{x^{2n}+n-1}{y^3+y^2+y}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^{2n}+n-1}{y^3+y^2+y}}=3\sqrt[3]{\prod\frac{x^{2n}+n-1}{x^3+x^2+x}}\overset{Lema}{\geq}3\sqrt[3]{\prod\frac{n}{3}}=n=RHS.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Problema261.

In non-obtse ΔABC

$$r+r_a\leq\sqrt{7b^2+7c^2-2bc-4a^2}.$$

RMM 11/2024,Dang Ngoc Minh, Vietnam

Soluție.

Lema.

In non-obtse ΔABC

$$r+r_a\leq b+c.$$

Demonstrație

$$r+r_a=\frac{S}{p}+\frac{S}{p-a}=S\left(\frac{1}{p}+\frac{1}{p-a}\right)=S\left(\frac{p-a+p}{p(p-a)}\right)=S\left(\frac{2p-a}{p(p-a)}\right)=pr\frac{b+c}{p(p-a)}=$$

$$=(b+c)\frac{r}{p-a}=(b+c)\tan\frac{A}{2}\overset{A\le\frac{\pi}{2}}{\le}(b+c)\cdot 1=b+c.$$

$$LHS=r+r_a\overset{Lema}{\le}b+c\overset{(1)}{\le}\sqrt{7b^2+7c^2-2bc-4a^2}=RHS,$$

unde $b+c\overset{(1)}{\le}\sqrt{7b^2+7c^2-2bc-4a^2}\Leftrightarrow(b+c)^2\le 7b^2+7c^2-2bc-4a^2\Leftrightarrow$

$$\Leftrightarrow 6b^2+6c^2-4bc-4a^2\ge 0\Leftrightarrow 4\left(b^2+c^2-a^2\right)+2(b-c)^2\ge 0,$$

care rezultă din $\left(b^2+c^2-a^2\right)\ge 0$ in non-obtse ΔABC.

Egalitatea are loc dacă și numai dacă $\left(b^2+c^2-a^2\right)=0$ și $(b-c)^2=0$, adică în triunghiul dreptunghic isoscel.

Remarca.

In non-obtse ΔABC

$$r+r_a\le\sqrt{2\left(b^2+c^2\right)}.$$

Marin Chirciu

Soluție.

Lema.

In non-obtse ΔABC

$$r+r_a\le b+c.$$

$$LHS=r+r_a\overset{Lema}{\le}b+c\overset{CBS}{\le}\sqrt{2\left(b^2+c^2\right)}=RHS.$$

Egalitatea are loc dacă și numai dacă ΔABC este dreptunghic isoscel.

Problema262.

If $a,b,c>0$, $abc=1$ then

$$\sum\frac{bc}{1+a}\ge\frac{27}{(a+b+c)(3+a+b+c)}.$$

Mathematical Inequalities 11/2024

Soluție.

$$\sum\frac{bc}{1+a}\geq\frac{27}{(a+b+c)(3+a+b+c)}\overset{abc=1}{\Leftrightarrow}\sum\frac{1}{a+a^2}\geq\frac{27}{(a+b+c)(3+a+b+c)}.$$

Funcția $f(x)=\frac{1}{x+x^2}$, $f'(x)=\frac{-2x-1}{(x+x^2)^2}$, $f''(x)=\frac{6x^2+6x+2}{(x+x^2)^3}>0\Rightarrow f$ este convexă.

Folosind inegalitatea lui Jensen obținem:

$$f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)=3\cdot\frac{1}{\frac{a+b+c}{3}+\left(\frac{a+b+c}{3}\right)^2}=\frac{27}{(a+b+c)(3+a+b+c)}.$$

Remarca.

If $a,b,c>0$, $abc=1$ and $\lambda\geq 0$ then

$$\sum\frac{bc}{1+\lambda a}\geq\frac{27}{(a+b+c)(3+\lambda(a+b+c))}.$$

Marin Chirciu

Soluție.

$$\sum\frac{bc}{1+\lambda a}\geq\frac{27}{(a+b+c)(3+a+b+c)}\overset{abc=1}{\Leftrightarrow}\sum\frac{1}{a+\lambda a^2}\geq\frac{27}{(a+b+c)(3+a+b+c)}.$$

Considerăm funcția $f(x)=\frac{1}{x+\lambda x^2}$, $f'(x)=\frac{-2\lambda x-1}{(x+x^2)^2}$, $f''(x)=\frac{6\lambda^2x^2+6\lambda x+2}{(x+x^2)^3}>0\Rightarrow$

f este convexă.

Folosind inegalitatea lui Jensen obținem:

$$f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)=3\cdot\frac{1}{\frac{a+b+c}{3}+\lambda\left(\frac{a+b+c}{3}\right)^2}=\frac{27}{3(a+b+c)+\lambda(a+b+c)^2}$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema263.

If $a_1,a_2,...,a_n,b_1,b_2,...,b_n>0$ then

$$\sum\frac{1}{a_1b_1}\sum(a_1+b_1)^2\geq 4n^2.$$

Mathematical Inequalities 11/2024, Imad Zak, Lebanon

Soluție.

$$LHS=\sum\frac{1}{a_1b_1}\sum(a_1+b_1)^2\overset{CBS}{\geq}\sum\frac{1}{a_1b_1}\sum4a_1b_1=4\sum\frac{1}{a_1b_1}\sum a_1b_1\overset{CS}{\geq}4(1+1+...+1)^2=$$

$$=4n^2=RHS.$$

Am folosit mai sus $(x+y)^2\geq4xy$, pentru $(x,y)=(a_i,b_i)$, $i=\overline{1,n}$.

Remarca.

If $a_1,a_2,...,a_n,b_1,b_2,...,b_n>0$ then

$$\sum\frac{1}{a_1b_1}\sum\left(a_1^2+b_1^2\right)\geq2n^2.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{1}{a_1b_1}\sum(a_1+b_1)^2\overset{CBS}{\geq}\sum\frac{1}{a_1b_1}\sum2a_1b_1=2\sum\frac{1}{a_1b_1}\sum a_1b_1\overset{CS}{\geq}2(1+1+...+1)^2=$$

$$=2n^2=RHS.$$

Am folosit mai sus $x^2+y^2\geq2xy$, pentru $(x,y)=(a_i,b_i)$, $i=\overline{1,n}$.

Egalitatea are loc dacă și numai dacă $a_i=b_i, i=\overline{1,n}$.

Problema264.

Solve for real numbers

$$\sqrt{x-y}+2\sqrt{y-z}+3\sqrt{z+x}=x+7.$$

Problem(108), Konstantinos Geronikolas, Greece

Soluție.

$$x+7=\sqrt{x-y}+2\sqrt{y-z}+3\sqrt{z+x}\overset{CBS}{\leq}\sqrt{\left(1^2+2^2+3^2\right)(x-y+y-z+z+x)}=\sqrt{14\cdot2x}=2\sqrt{7x}.$$

Din $x+7\leq2\sqrt{7x}\Leftrightarrow\left(\sqrt{x}-\sqrt{7}\right)^2\leq0\Leftrightarrow x=7$.

Egalitatea în CBS are loc dacă și numai dacă

$$\frac{1}{x-y}=\frac{4}{y-z}=\frac{9}{z+x}=\frac{1+4+9}{x-y+y-z+z+x}=\frac{14}{2x}=\frac{7}{x}.$$

Din $\frac{1}{x-y}=\frac{7}{x}$ și $x=7 \Rightarrow y=6$.

Din $\frac{9}{z+x}=\frac{7}{x}$ și $x=7 \Rightarrow z=2$.

Soluția ecuației este $(x,y,z)=(7,6,2)$.

Remarca.

Let $\lambda>0$ fixed. Solve for real numbers

$$\sqrt{x-y}+\lambda\sqrt{y-z}+(\lambda+1)\sqrt{z+x}=x+\lambda^2+\lambda+1.$$

Marin Chirciu

Soluție.

$$x+\lambda^2+\lambda+1=\sqrt{x-y}+\lambda\sqrt{y-z}+(\lambda+1)\sqrt{z+x}\overset{CBS}{\leq}\sqrt{\left(1^2+\lambda^2+(\lambda+1)^2\right)(x-y+y-z+z+x)}=$$
$$=\sqrt{(2\lambda^2+2\lambda+2)\cdot 2x}=2\sqrt{(\lambda^2+\lambda+1)x}.$$

Din $x+\lambda^2+\lambda+1\leq 2\sqrt{x(\lambda^2+\lambda+1)} \Leftrightarrow \left(\sqrt{x}-\sqrt{\lambda^2+\lambda+1}\right)^2\leq 0 \Leftrightarrow x=\lambda^2+\lambda+1$.

Egalitatea în CBS are loc dacă și numai dacă

$$\frac{1}{x-y}=\frac{\lambda^2}{y-z}=\frac{(\lambda+1)^2}{z+x}=\frac{1+\lambda^2+(\lambda+1)^2}{x-y+y-z+z+x}=\frac{2(\lambda^2+\lambda+1)}{2x}=\frac{\lambda^2+\lambda+1}{x}.$$

Din $\frac{1}{x-y}=\frac{\lambda^2+\lambda+1}{x}$ și $x=\lambda^2+\lambda+1 \Rightarrow y=\lambda^2+\lambda$.

Din $\frac{(\lambda+1)^2}{z+x}=\frac{7}{x}$ și $x=\lambda^2+\lambda+1 \Rightarrow z=\lambda$.

Soluția ecuației este $(x,y,z)=(\lambda^2+\lambda+1,\lambda^2+\lambda,\lambda)$.

Cazul $\lambda=3$.

Solve for real numbers

$$\sqrt{x-y}+3\sqrt{y-z}+4\sqrt{z+x}=x+13.$$

Soluția ecuației este $(x,y,z)=(13,12,3)$.

Problema265.

If $a,b,c>0$, $a^2+b^2+c^2=1$ then

$$\sum a\sqrt{a(b+c)}\le\sqrt{2}.$$

MathAtelier 11/2024, Panagiotis Danousis, Greece

Soluție.

$$LHS=\sum a\sqrt{a(b+c)}\overset{CBS}{\le}\sqrt{\sum a^2\sum a(b+c)}=\sqrt{1\cdot 2\sum ab}\overset{SOS}{\le}\sqrt{2\sum a^2}=\sqrt{2\cdot 1}=\sqrt{2}=RHS.$$

Remarca.

If $a,b,c>0$, $a^2+b^2+c^2=1$ and $\lambda\ge 0$ then

$$\sum a\sqrt{a(b+\lambda c)}\le\sqrt{\lambda+1}.$$

Marin Chirciu

Soluție.

$$LHS=\sum a\sqrt{a(b+\lambda c)}\overset{CBS}{\le}\sqrt{\sum a^2\sum a(b+\lambda c)}=\sqrt{1\cdot(\lambda+1)\sum ab}\overset{SOS}{\le}\sqrt{(\lambda+1)\sum a^2}=$$

$$=\sqrt{(\lambda+1)\cdot 1}=\sqrt{\lambda+1}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=\frac{1}{\sqrt{3}}$.

Problema266.

In ΔABC

$$\sum\frac{bc}{w_a}\le 6R.$$

RMM 11/2024, Nguyen Minh Tho, Vietnam

Soluție

$$LHS=\sum\frac{bc}{w_a}\le\sum\frac{bc}{h_a}=\sum\frac{bc}{\frac{2S}{a}}=\sum\frac{abc}{2S}=\sum\frac{4RS}{2S}=\sum 2R=6R=RHS.$$

Remarca.

In ΔABC

$$1)\quad \sum \frac{bc}{m_a} \le 6R.$$

$$2)\quad \sum \frac{bc}{s_a} \le 6R.$$

$$3)\ 12r \le \sum \frac{bc}{r_a} \le \frac{3R^3}{2r^2}.$$

Soluție

$$\sum \frac{bc}{r_a} = \sum \frac{bc}{\frac{S}{p-a}} = \sum \frac{bc(p-a)}{S} = \frac{p(p^2+r^2-8Rr)}{pr} = \frac{p^2+r^2-8Rr}{r} \overset{Gerretsen}{\le} \frac{3R^3}{2r^2}.$$

$$\sum \frac{bc}{r_a} = \frac{p^2+r^2-8Rr}{r} \overset{Gerretsen}{\ge} 12r.$$

$$4)\ \sum \frac{a(b+c)}{w_a} \le \frac{6R^2}{r}.$$

$$5)\ \sum \frac{a(b+c)}{m_a} \le \frac{6R^2}{r}.$$

$$6)\ \sum \frac{a(b+c)}{s_a} \le \frac{6R^2}{r}.$$

$$7)\ \sum \frac{a(b+c)}{r_a} = 12R.$$

$$8)\ \sum \frac{a(b+c)}{w_b+w_c} \le 6R.$$

$$9)\ \sum \frac{a(b+c)}{m_b+m_c} \le 6R.$$

$$10)\ \sum \frac{a(b+c)}{r_b+r_c} = 4(R+r).$$

$$11)\ \sum \frac{a(b+c)}{m_b+m_c} \le 6R.$$

12) $\sum \frac{bc}{w_b + w_c} \le \frac{3R^4}{8r^3}$.

Soluție

$$LHS = \sum \frac{bc}{w_b + w_c} \le \sum \frac{bc}{h_b + h_c} \le \frac{3R^4}{8r^3}$$

13) $\sum \frac{bc}{s_b + s_c} \le \frac{3R^4}{8r^3}$.

Soluție

$$LHS = \sum \frac{bc}{s_b + s_c} \le \sum \frac{bc}{h_b + h_c} \le \frac{3R^4}{8r^3}.$$

14) $\frac{12r^2}{R} \le \sum \frac{bc}{h_b + h_c} \le \frac{3R^4}{8r^3}$.

Soluție

$$\sum \frac{bc}{h_b + h_c} = \frac{p^6 + p^4\left(3r^2 - 4Rr\right) + p^2r^2\left(32R^2 + 8Rr + 3r^2\right) + r^3\left(4R + r\right)^3}{4rp^2\left(p^2 + r^2 + 2Rr\right)}.$$

Inegalitatea din dreapta.

$$\sum \frac{bc}{h_b + h_c} = \frac{p^6 + p^4\left(3r^2 - 4Rr\right) + p^2r^2\left(32R^2 + 8Rr + 3r^2\right) + r^3\left(4R + r\right)^3}{4rp^2\left(p^2 + r^2 + 2Rr\right)} \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le} \frac{8R^6 + 16R^5r + 48R^4r^2 + 64R^3r^3 + 25R^2r^4 + 28Rr^5 + 8r^6}{r^3\left(144R^2 - 77Rr + 10r^2\right)} \overset{Euler}{\le} \frac{\frac{81R^6}{2}}{r^3 \cdot 108R^2} = \frac{3R^4}{8r^3} = RHS.$$

Inegalitatea din stânga.

$$\sum \frac{bc}{h_b + h_c} = \frac{p^6 + p^4\left(3r^2 - 4Rr\right) + p^2r^2\left(32R^2 + 8Rr + 3r^2\right) + r^3\left(4R + r\right)^3}{4rp^2\left(p^2 + r^2 + 2Rr\right)} \overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} \frac{8r^2\left(228R^3 - 151R^2r + 40Rr^2 - 4r^3\right)}{27R^2\left(2R^2 + 3Rr + 2r^2\right)} \overset{Euler}{\ge} \frac{8r^2 \cdot 162R^3}{27R^2 \cdot 4R^2} = \frac{12r^2}{R}$$

15) $6r \le \sum \frac{bc}{r_b + r_c} \le 3R$.

Dezvoltări, Marin Chirciu

Soluție

$$\sum \frac{bc}{r_b + r_c} = \frac{p^2\left(p^2 + 2r^2 - 4Rr\right) + r\left(4R + r\right)^3}{4Rp^2}.$$

Demonstrație

Inegalitatea din dreapta.

$$\sum \frac{bc}{r_b + r_c} = \frac{p^2\left(p^2 + 2r^2 - 4Rr\right) + r\left(4R + r\right)^3}{4Rp^2} = \frac{1}{4R}\left[p^2 + 2r^2 - 4Rr + \frac{r\left(4R + r\right)^3}{p^2}\right] \overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq} \frac{1}{4R}\left[4R^2 + 4Rr + 3r^2 + 2r^2 - 4Rr + \frac{r\left(4R + r\right)^3}{\frac{r\left(4R + r\right)^2}{R + r}}\right] = \frac{1}{4R}\left[4R^2 + 5r^2 + \left(4R + r\right)\left(R + r\right)\right] =$$

$$= \frac{8R^2 + 5Rr + 6r^2}{4R} \overset{Euler}{\leq} \frac{12R^2}{4R} = 3R.$$

Inegalitatea din stânga.

$$\sum \frac{bc}{r_b + r_c} = \frac{p^2\left(p^2 + 2r^2 - 4Rr\right) + r\left(4R + r\right)^3}{4Rp^2} = \frac{1}{4R}\left[p^2 + 2r^2 - 4Rr + \frac{r\left(4R + r\right)^3}{p^2}\right] \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{1}{4R}\left[16Rr - 5r^2 + 2r^2 - 4Rr + \frac{r\left(4R + r\right)^3}{\frac{R\left(4R + r\right)^2}{2\left(2R - r\right)}}\right] = \frac{1}{4R}\left[12Rr - 3r^2 + \frac{2\left(2R - r\right)r\left(4R + r\right)}{R}\right] =$$

$$= \frac{r}{4R} \cdot \frac{12R^2 - 3Rr + 2\left(2R - r\right)\left(4R + r\right)}{R} = \frac{r\left(28R^2 - 7Rr - 2r^2\right)}{4R^2} \overset{Euler}{\geq} \frac{r \cdot 24R^2}{4R^2} = 6r.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema267.

If $a, b, c > 0$, $abc = 1$ then

$$\sum \frac{a^4}{a^2 + bc} \geq \frac{3}{2}.$$

RMM 11/2024, Problem(134), Konstantinos Geronikolas, Greece

Soluție

$$LHS=\sum\frac{a^4}{a^2+bc}\overset{CBS}{\geq}\frac{\left(\sum a^2\right)^2}{\sum\left(a^2+bc\right)}=\frac{\left(\sum a^2\right)^2}{\sum a^2+\sum bc}\overset{SOS}{\geq}\frac{\left(\sum a^2\right)^2}{\sum a^2+\sum a^2}=\frac{1}{2}\sum a^2\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{1}{2}\cdot 3\sqrt[3]{a^2b^2c^2}=\frac{3}{2}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc=1$ and $\lambda\geq 0$ then

$$\sum\frac{a^4}{a^2+\lambda bc}\geq\frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{a^4}{a^2+\lambda bc}\overset{CBS}{\geq}\frac{\left(\sum a^2\right)^2}{\sum\left(a^2+\lambda bc\right)}=\frac{\left(\sum a^2\right)^2}{\sum a^2+\lambda\sum bc}\overset{SOS}{\geq}\frac{\left(\sum a^2\right)^2}{\sum a^2+\lambda\sum a^2}=\frac{1}{\lambda+1}\sum a^2\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{1}{\lambda+1}\cdot 3\sqrt[3]{a^2b^2c^2}=\frac{3}{\lambda+1}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema268.

In ΔABC

$$\sum\frac{m_a}{a^2+bc}\geq\frac{3}{4R}.$$

RMM 11/2024,Nguyen Minh Tho, Vietnam

Soluție

$$LHS=\sum\frac{m_a}{a^2+bc}\overset{Tereshin}{\geq}\sum\frac{\frac{b^2+c^2}{4R}}{a^2+bc}=\frac{1}{4R}\sum\frac{b^2+c^2}{a^2+bc}\overset{CBS}{\geq}\frac{1}{4R}\sum\frac{b^2+c^2}{\sqrt{\left(a^2+b^2\right)\left(a^2+c^2\right)}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{1}{4R}\cdot 3\sqrt[3]{\prod\frac{b^2+c^2}{\sqrt{\left(a^2+b^2\right)\left(a^2+c^2\right)}}}=\frac{3}{4R}=RHS.$$

Remarca.

In ΔABC

$$\sum \frac{m_b + m_c}{a^2 + bc} \geq \frac{3}{2R}.$$

Marin Chirciu

Soluție

$$LHS = \sum \frac{m_b + m_c}{a^2 + bc} \overset{Tereshin}{\geq} \sum \frac{\frac{a^2+c^2}{4R} + \frac{a^2+b^2}{4R}}{a^2 + bc} = \frac{1}{4R} \sum \frac{(a^2+b^2)+(a^2+c^2)}{a^2+bc} \overset{AM-GM\,\&CBS}{\geq}$$

$$\overset{AM-GM\,\&CBS}{\geq} \frac{1}{4R} \sum \frac{2\sqrt{(a^2+b^2)(a^2+c^2)}}{\sqrt{(a^2+b^2)(a^2+c^2)}} = \frac{1}{4R} \sum 2 = \frac{6}{4R} = \frac{3}{2R} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema269.

In ΔABC

$$\sum m_a \cot \frac{A}{2} \geq \frac{2p^3}{9Rr}.$$

RMM 11/2024, Nguyen Minh Tho, Vietnam

Soluție

$$LHS = \sum m_a \cot \frac{A}{2} \overset{Tereshin}{\geq} \sum \frac{b^2+c^2}{4R} \sqrt{\frac{p(p-a)}{(p-b)(p-c)}} = \sum \frac{b^2+c^2}{4R} \frac{p(p-a)}{\sqrt{p(p-a)(p-b)(p-c)}} =$$

$$= \sum \frac{b^2+c^2}{4R} \frac{p(p-a)}{S} = \frac{p}{4RS} \sum (b^2+c^2)(p-a) = \frac{p}{4Rrp} \cdot 2p(p^2 - 3r^2 - 6Rr) =$$

$$= \frac{p(p^2 - 3r^2 - 6Rr)}{2Rr} \overset{Gerretsen}{\geq} \frac{2p^3}{9Rr} = RHS,$$

Remarca.

In ΔABC

$$1) \sum m_a \tan \frac{A}{2} \geq \frac{27r^2}{p}.$$

Marin Chirciu

Soluție

$$LHS=\sum m_a\tan\frac{A}{2}\overset{Tereshin}{\geq}\sum\frac{b^2+c^2}{4R}\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\sum\frac{b^2+c^2}{4R}\frac{(p-b)(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}}=$$

$$=\sum\frac{b^2+c^2}{4R}\frac{(p-b)(p-c)}{S}=\frac{1}{4RS}\sum(b^2+c^2)(p-b)(p-c)=\frac{1}{4Rrp}\cdot 2r\left[p^2(2R+3r)-r(4R+r)^2\right]=$$

$$\frac{p^2(2R+3r)-r(4R+r)^2}{2Rp}\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(2R+3r)-r(4R+r)^2}{2Rp}=$$

$$=\frac{r(16R^2+30Rr-16r^2)}{2Rp}=\frac{r(8R^2+15Rr-8r^2)}{Rp}\overset{Euler}{\geq}\frac{r\cdot 27Rr}{Rp}=\frac{27r^2}{p}=RHS.$$

$$2)\sum m_a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\geq\frac{54Rr}{p}.$$

Marin Chirciu

Soluție

$$1)\sum m_a\tan\frac{A}{2}\geq\frac{27r^2}{p}.$$

$$\sum m_a\tan\frac{A}{2}\overset{Tereshin}{\geq}\sum\frac{b^2+c^2}{4R}\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\sum\frac{b^2+c^2}{4R}\frac{(p-b)(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}}=$$

$$=\sum\frac{b^2+c^2}{4R}\frac{(p-b)(p-c)}{S}=\frac{1}{4RS}\sum(b^2+c^2)(p-b)(p-c)=\frac{1}{4Rrp}\cdot 2r\left[p^2(2R+3r)-r(4R+r)^2\right]=$$

$$\frac{p^2(2R+3r)-r(4R+r)^2}{2Rp}\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(2R+3r)-r(4R+r)^2}{2Rp}\overset{Euler}{\geq}\frac{27r^2}{p}$$

$$2)\sum m_a\cot\frac{A}{2}\geq\frac{p(5R-4r)}{R}$$

$$\sum m_a\cot\frac{A}{2}\overset{Tereshin}{\geq}\sum\frac{b^2+c^2}{4R}\sqrt{\frac{p(p-a)}{(p-b)(p-c)}}=\sum\frac{b^2+c^2}{4R}\frac{p(p-a)}{\sqrt{p(p-a)(p-b)(p-c)}}=$$

$$=\sum\frac{b^2+c^2}{4R}\frac{p(p-a)}{S}=\frac{p}{4RS}\sum(b^2+c^2)(p-a)=\frac{p}{4Rrp}\cdot 2p\left(p^2-3r^2-6Rr\right)=$$

$$=\frac{p\left(p^2-3r^2-6Rr\right)}{2Rr}\overset{Gerretsen}{\geq}\frac{p\left(16Rr-5r^2-3r^2-6Rr\right)}{2Rr}\overset{Euler}{\geq}\frac{54Rr}{p}.$$

Adunând inegalitățile 1) și 2) obținem $\sum m_a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\geq\frac{27r^2}{p}+\frac{p(5R-4r)}{R}=$

$$=\frac{p^2(5R-4r)+27Rr^2}{pR}\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(5R-4r)+27Rr^2}{pR}=$$

$$=\frac{2r(40R^2-31Rr+10r^2)}{pR}\overset{Euler}{\geq}\frac{2r\cdot 27R^2}{pR}=\frac{54Rr}{p}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema270.

Fie șirul $(a_n)_{n\geq 1}$ definit prin $a_1=1$ și $a_{n+1}=\frac{9a_n+4}{a_n+6}$, $n\geq 1$. Arătați că șirul este convergent și calulați limita.

Bulgarian Mathematical Competition-2023

Soluție

Demonstrăm prin inducție matematică : $a_n=\frac{4\cdot 2^n-3}{2^n+3}$, $n\geq 1$.

Șirul cu termenul general $a_n=\frac{4\cdot 2^n-\lambda+1}{2^n+\lambda-1}$, $n\geq 1$ este convergent și are limita λ .

Remarca.

Fie $\lambda>0$ fixat. Fie șirul $(a_n)_{n\geq 1}$ definit prin $a_1=1$ și $a_{n+1}=\frac{(2\lambda+1)a_n+\lambda}{a_n+\lambda+2}$, $n\geq 1$. Arătați că șirul este convergent și calulați limita.

Marin Chirciu

Soluție

Demonstrăm prin inducție matematică : $a_n=\frac{\lambda\cdot 2^n-\lambda+1}{2^n+\lambda-1}$, $n\geq 1$.

Șirul cu termenul general $a_n=\frac{\lambda\cdot 2^n-\lambda+1}{2^n+\lambda-1}$, $n\geq 1$ este convergent și are limita egalî cu 4.

Problema271.

Find A_{2024}, where

$$A_n = 1\cdot 2 + 3\cdot 4 + 5\cdot 8 + \ldots + (2n-1)\cdot 2^n.$$

Bulgarian Mathematical Competition-2023,Nedialka Dumitrova

Soluție

$2A_n = 1\cdot 4 + 3\cdot 8 + 5\cdot 16 + \ldots + (2n-3)\cdot 2^n + (2n+1)\cdot 2^{n+1}$.

$A_n = 2A_n - A_n = (2n-1)\cdot 2^{n+1} - (1\cdot 2 + 2\cdot 4 + 2\cdot 8 + \ldots + 2\cdot 2^n) =$

$= (2n-1)\cdot 2^{n+1} - 2(2+4+8+\ldots+2^n) + 1\cdot 2 = (2n-1)\cdot 2^{n+1} - 2\dfrac{2^{n+1}-2}{2-1} + 2 =$

$= (2n-1)\cdot 2^{n+1} - 2^{n+2} + 4 + 2 = 2^{n+1}(2n-1-2) + 6 = 2^{n+1}(2n-3) + 6$.

Deducem că $A_{2024} = 2^{2025}(4048-3) + 6 = 2^{2025}\cdot 4045 + 6$.

Remarca.

Find A_{2025}, where

$$A_n = 2\cdot 3 + 5\cdot 9 + 8\cdot 27 + \ldots + (3n-1)\cdot 3^n,\ n \in \mathbf{N}^*.$$

Marin Chirciu

Soluție

$3A_n = 2\cdot 3^2 + 5\cdot 3^3 + 8\cdot 3^4 + \ldots + (3n-4)\cdot 3^n + (3n-1)\cdot 3^{n+1}$.

$2A_n = 3A_n - A_n = (3n-1)\cdot 3^{n+1} - 3^3 - 3^4 - 3^5 - \ldots - 3^{n+1} - 2\cdot 3 = (3n-1)\cdot 3^{n+1} - S_n - 6$.

Avem $S_n = 3^3 + 3^4 + 3^5 + \ldots + 3^{n+1} = 3^3\dfrac{3^{n-1}-1}{2} = \dfrac{3^{n+2}-27}{2}$.

Obținem:

$2A_n = (3n-1)\cdot 3^{n+1} - S_n - 6 = (3n-1)\cdot 3^{n+1} - \dfrac{3^{n+2}-27}{2} - 6 = (3n-1)\cdot 3^{n+1} - \dfrac{3^{n+2}}{2} + \dfrac{15}{2} =$

$= \dfrac{1}{2}\left[2(3n-1)\cdot 3^{n+1} - 3^{n+2} + 15\right] = \dfrac{1}{2}\left[3^{n+1}(6n-5) + 15\right]$.

$2A_n = \dfrac{1}{2}\left[3^{n+1}(6n-5) + 15\right] \Rightarrow A_n = \dfrac{1}{4}\left[3^{n+1}(6n-5) + 15\right]$.

Deducem că $A_{2025} = \dfrac{1}{4}\left[3^{2026}(6\cdot 2025 - 5) + 15\right] = \dfrac{1}{4}\left(12145\cdot 3^{2026} + 15\right)$.

Problema272.

In ΔABC

$$\sum \frac{1}{w_a} \geq \frac{2\sqrt{3}}{\sqrt{p^2+r^2-8Rr}}.$$

RMM 11/2024, Nguyen Minh Tho, Vietnam

Soluție

$$\sum \frac{1}{w_a} \overset{CS}{\geq} \frac{9}{\sum w_a} \overset{w_a \leq \sqrt{p(p-a)}}{\geq} \frac{9}{\sum \sqrt{p(p-a)}} \overset{CBS}{\geq} \frac{9}{p\sqrt{3}} \overset{Gerretsen}{\geq} \frac{2\sqrt{3}}{\sqrt{p^2+r^2-8Rr}}.$$

Remarca.

In ΔABC

1) $$\frac{4}{R} \leq \sum \frac{1}{w_a \sin \frac{A}{2}} \leq \frac{2}{r}.$$

Soluție

$$\sum \frac{1}{w_a \sin \frac{A}{2}} = \frac{2}{r}, \text{(Gergonne)}.$$

2) $$\sum \frac{1}{\left(w_a \sin \frac{A}{2}\right)^n} \geq 3\left(\frac{2}{3r}\right)^n, n \in \mathbf{N}.$$

Dezvoltări, Marin Chirciu

Soluție

Pentru $n=0$ se obține egalitatea3=3.

Pentru $n=1$ se obține $\sum \frac{1}{w_a \sin \frac{A}{2}} = \frac{2}{r}$,(Gergonne).

Pentru $n \geq 2$ se folosește inegalitatea lui Holder.

$$LHS=\sum\frac{1}{\left(w_a\sin\frac{A}{2}\right)^n}=\sum\left(\frac{1}{w_a\sin\frac{A}{2}}\right)^n\overset{Holder}{\geq}\frac{\left(\sum\frac{1}{w_a\sin\frac{A}{2}}\right)^n}{3^{n-1}}\overset{Gergonne}{=}\frac{\left(\frac{2}{r}\right)^n}{3^{n-1}}=3\left(\frac{2}{3r}\right)^n=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema273.

In ΔABC

$$\frac{9R}{a^2+b^2+c^2}\leq\sum\frac{1}{h_a+\sqrt{h_bh_c}}\leq\frac{1}{2r}.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Inegalitatea din dreapta.

$$\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{CS}{\leq}\sum\frac{1}{4}\left(\frac{1}{h_a}+\frac{1}{\sqrt{h_bh_c}}\right)=\frac{1}{4}\sum\left(\frac{1}{h_a}+\sqrt{\frac{1}{h_b}\cdot\frac{1}{h_c}}\right)\overset{AM-GM}{\leq}\frac{1}{4}\sum\left(\frac{1}{h_a}+\frac{1}{2}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)=$$

$$=\frac{1}{4}\cdot2\sum\frac{1}{h_a}=\frac{1}{2}\cdot\frac{1}{r}=\frac{1}{2r}.$$

Inegalitatea din stânga.

$$\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{CS}{\geq}\frac{9}{\sum\left(h_a+\sqrt{h_bh_c}\right)}\overset{AM-GM}{\geq}\frac{9}{\sum\left(h_a+\frac{h_b+h_c}{2}\right)}=\frac{9}{2\sum h_a}\overset{(1)}{\geq}\frac{9R}{\sum a^2},$$

unde $\frac{9}{2\sum h_a}\overset{(1)}{\geq}\frac{9R}{\sum a^2}\Leftrightarrow\sum h_a\leq\frac{\sum a^2}{2R}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ \frac{9R}{4(R+r)^2}\leq\sum\frac{1}{h_a+\sqrt{h_bh_c}}\leq\frac{1}{2r}.$$

$$2)\ \frac{1}{R} \le \sum \frac{1}{r_a + \sqrt{r_b r_c}} \le \frac{1}{2r}.$$

$$3)\ \frac{1}{R} \le \sum \frac{1}{m_a + \sqrt{m_b m_c}} \le \frac{1}{2r}.$$

$$4)\ \frac{1}{R} \le \sum \frac{1}{w_a + \sqrt{w_b w_c}} \le \frac{1}{2r}.$$

$$5)\ \frac{1}{R} \le \sum \frac{1}{s_a + \sqrt{s_b s_c}} \le \frac{1}{2r}.$$

Dezvoltări, Marin Chirciu

Problema274.

If $a, b > 0$ then

$$\sqrt{a^2 + 8b^2} + \sqrt{b^2 + 8a^2} + \frac{a+1}{b^2} + \frac{b+1}{a^2} \ge 10.$$

RMM 11/2024,Nguyen Hung Cuong, Vietnam

Soluție

Lema.

If $a, b > 0$ then

$$\sqrt{a^2 + 8b^2} \ge \frac{a+8b}{3}.$$

Demonstrație

$$\sqrt{a^2 + 8b^2} \ge \frac{a+8b}{3} \Leftrightarrow 3\sqrt{a^2 + 8b^2} \ge (a+8b) \Leftrightarrow 9(a^2 + 8b^2) \ge (a+8b)^2 \Leftrightarrow 8(a-b)^2 \ge 0.$$

$$LHS = \sqrt{a^2 + 8b^2} + \sqrt{b^2 + 8a^2} + \frac{a+1}{b^2} + \frac{b+1}{a^2} \overset{Lema}{\ge} \frac{a+8b}{3} + \frac{b+8a}{3} + \frac{a^3 + b^3 + a^2 + b^2}{a^2 b^2} =$$

$$3a + 3b + \frac{a^3 + b^3}{a^2 b^2} + \frac{a^2 + b^2}{a^2 b^2} = (a+b) + (a+b) + (a+b) + \frac{a^3 + b^3}{a^2 b^2} + \frac{a^2 + b^2}{a^2 b^2} \overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge} 5\sqrt[5]{(a+b)\cdot(a+b)\cdot(a+b)\cdot\frac{a^3 + b^3}{a^2 b^2}\cdot\frac{a^2 + b^2}{a^2 b^2}} = 5\sqrt[5]{(a+b)^3\cdot\frac{a^3 + b^3}{a^2 b^2}\cdot\frac{a^2 + b^2}{a^2 b^2}} \overset{(1)}{\ge} 10 = RHS,$$

unde $5\sqrt[5]{(a+b)^3\cdot\frac{a^3+b^3}{a^2b^2}\cdot\frac{a^2+b^2}{a^2b^2}}\overset{(1)}{\geq}10 \Leftrightarrow \sqrt[5]{(a+b)^3\cdot\frac{a^3+b^3}{a^2b^2}\cdot\frac{a^2+b^2}{a^2b^2}}\geq 2 \Leftrightarrow$

$\Leftrightarrow (a+b)^3\cdot\frac{a^3+b^3}{a^2b^2}\cdot\frac{a^2+b^2}{a^2b^2}\geq 32 \Leftrightarrow (a+b)^3(a^3+b^3)(a^2+b^2)\geq 32a^4b^4$, care rezultă din AM-GM: $a+b\geq 2\sqrt{ab}$, $a^3+b^3\geq 2\sqrt{a^3b^3}$ și $a^2+b^2\geq 2ab$.

Obținem:

$$(a+b)^3(a^3+b^3)(a^2+b^2)\overset{AM-GM}{\geq}\left(2\sqrt{ab}\right)^3\cdot 2\sqrt{a^3b^3}\cdot 2ab=32a^4b^4.$$

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

If $a,b>0$ and $k\in\mathbf{N}^*$ then

$$\sqrt{a^2+4k(k-1)b^2}+\sqrt{b^2+4k(k-1)a^2}+\frac{a+1}{b^k}+\frac{b+1}{a^k}\geq 2(2k+1).$$

Marin Chirciu

Soluție

Lema.

If $a,b>0$ and $\lambda>0$ then

$$\sqrt{a^2+\lambda b^2}\geq\frac{a+\lambda b}{\sqrt{\lambda+1}}.$$

Demonstrație

$$\sqrt{a^2+\lambda b^2}\geq\frac{a+\lambda b}{\sqrt{\lambda+1}}\Leftrightarrow\sqrt{\lambda+1}\sqrt{a^2+\lambda b^2}\geq a+\lambda b\Leftrightarrow(\lambda+1)(a^2+\lambda b^2)\geq(a+\lambda b)^2\Leftrightarrow$$

$$\Leftrightarrow \lambda(a-b)^2\geq 0.$$

Punem $\lambda=4k(k-1)\Rightarrow\sqrt{a^2+4k(k-1)b^2}\geq\frac{a+4k(k-1)b}{2k-1}$.

$$LHS=\sqrt{a^2+4k(k-1)b^2}+\sqrt{b^2+4k(k-1)a^2}+\frac{a+1}{b^k}+\frac{b+1}{a^k}\overset{Lema}{\geq}$$

$$\overset{Lema}{\geq}\frac{a+4k(k-1)b}{2k-1}+\frac{b+4k(k-1)a}{2k-1}+\frac{a^{k+1}+b^{k+1}}{a^kb^k}+\frac{a^k+b^k}{a^kb^k}=$$

$$=\frac{(2k-1)^2(a+b)}{2k-1}+\frac{a^{k+1}+b^{k+1}}{a^kb^k}+\frac{a^k+b^k}{a^kb^k}=(2k-1)(a+b)+\frac{a^{k+1}+b^{k+1}}{a^kb^k}+\frac{a^k+b^k}{a^kb^k}=$$

$$=\underbrace{(a+b)+...+(a+b)}_{2k-1}+\frac{a^{k+1}+b^{k+1}}{a^kb^k}+\frac{a^k+b^k}{a^kb^k}\overset{AM-GM}{\geq}(2k+1)\sqrt[2k+1]{(a+b)^{2k-1}\frac{a^{k+1}+b^{k+1}}{a^kb^k}\frac{a^k+b^k}{a^kb^k}}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}2(2k+1)=RHS,$$

unde $(2k+1)\sqrt[2k+1]{(a+b)^{2k-1}\frac{a^{k+1}+b^{k+1}}{a^kb^k}\frac{a^k+b^k}{a^kb^k}}\overset{(1)}{\geq}2(2k+1)\Leftrightarrow$

$$\sqrt[2k+1]{(a+b)^{2k-1}\frac{a^{k+1}+b^{k+1}}{a^kb^k}\frac{a^k+b^k}{a^kb^k}}\geq 2\Leftrightarrow (a+b)^{2k-1}\frac{a^{k+1}+b^{k+1}}{a^kb^k}\frac{a^k+b^k}{a^kb^k}\geq 2^{2k+1}\Leftrightarrow$$

$$\Leftrightarrow (a+b)^{2k-1}\left(a^{k+1}+b^{k+1}\right)\left(a^k+b^k\right)\geq 2^{2k+1}a^{2k}b^{2k},$$

care rezultă din AM-GM: $a+b\geq 2\sqrt{ab}$, $a^{k+1}+b^{k+1}\geq 2\sqrt{a^{k+1}b^{k+1}}$ și $a^k+b^k\geq 2\sqrt{a^kb^k}$.

Obținem:

$$(a+b)^{2k-1}\left(a^{k+1}+b^{k+1}\right)\left(a^k+b^k\right)\overset{AM-GM}{\geq}\left(2\sqrt{ab}\right)^{2k-1}\cdot 2\sqrt{a^{k+1}b^{k+1}}\cdot 2\sqrt{a^kb^k}=2^{2k+1}a^{2k}b^{2k}.$$

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

If $a,b>0$ then

1) $\sqrt{a^2+24b^2}+\sqrt{b^2+24a^2}+\frac{a+1}{b^3}+\frac{b+1}{a^3}\geq 14$.

2) $\sqrt{a^2+48b^2}+\sqrt{b^2+48a^2}+\frac{a+1}{b^4}+\frac{b+1}{a^4}\geq 18$.

Problema275.

In acute ΔABC

$$\frac{\sum a\sec A}{\prod\tan A}=2R.$$

Math Olymp 11/2024, Elton Papanikolla

Soluție

Folosim $\sum \tan A = \prod \tan A = \frac{2pr}{p^2-(2R+r)^2}$.

$$LHS = \frac{\sum a\sec A}{\prod \tan A} = \frac{\sum 2R\sin A\frac{1}{\cos A}}{\prod \tan A} = \frac{\sum 2R\tan A}{\prod \tan A} = \frac{2R\sum \tan A}{\prod \tan A} = 2R = RHS.$$

Remarca.

In acute ΔABC

$$4r \le \frac{\sum a\sec A}{\prod \tan A} \le 2R.$$

Marin Chirciu

Soluție

Folosim $\sum \tan A = \prod \tan A = \frac{2pr}{p^2-(2R+r)^2}$.

Inegalitatea din dreapta.

$$\frac{\sum a\sec A}{\prod \tan A} = \frac{\sum 2R\sin A\frac{1}{\cos A}}{\prod \tan A} = \frac{\sum 2R\tan A}{\prod \tan A} = \frac{2R\sum \tan A}{\prod \tan A} = 2R.$$

Inegalitatea din stânga.

$$\frac{\sum a\sec A}{\prod \tan A} = \frac{\sum 2R\sin A\frac{1}{\cos A}}{\prod \tan A} = \frac{\sum 2R\tan A}{\prod \tan A} = \frac{2R\sum \tan A}{\prod \tan A} = 2R \overset{Euler}{\ge} 4r.$$

Egalitatatea are loc dacă și numai dacă triunghiul este echilateral.

Problema276.

Determine all non-negative integer pairs (x, y) for which:

$$(xy-7)^2 = x^2 + y^2.$$

MathOlymp 11/2024, Mathew Bailla

Soluție

If $x \ge y \ge 4 \Rightarrow LHS = (xy-7)^2 \ge (4x-7)^2 > (2x)^2 > 2x^2 \ge x^2 + y^2 = RHS \Rightarrow$ fals.

If $x \geq y = 3 \Rightarrow LHS = (xy-7)^2 \geq (3x-7)^2 = x^2 + 3^2 = RHS \Rightarrow (3x-7)^2 = x^2 + 9 \Leftrightarrow$

$\Leftrightarrow 4x^2 - 21x + 20 = 0 \Leftrightarrow (x-4)(4x-5) = 0 \Leftrightarrow x = 4 \Rightarrow (x,y) = (4,3)$.

If $x \geq y, 1 \leq y \leq 2 \Rightarrow$ fals.

If $x \geq y = 0 \Rightarrow x = 7 \Rightarrow (x,y) = (7,0)$.

Deducem că ecuația admite soluțiile $(x,y) = (4,3),(3,4),(7,0),(0,7)$.

Problema277.

In ΔABC

$$\sum \frac{a^2}{\cos^2 \frac{A}{2}} \geq 12R^2.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Lema.

In ΔABC

$$\sum \frac{a^2}{\cos^2 \frac{A}{2}} = 8R(2R-r).$$

$$LHS = \sum \frac{a^2}{\cos^2 \frac{A}{2}} = 8R(2R-r) \overset{Euler}{\geq} 12R^2 = RHS.$$

Remarca.

In ΔABC

$$1)\ 12R^2 \leq \sum \frac{a^2}{\cos^2 \frac{A}{2}} \leq \frac{6R^3}{r}.$$

.

$$2)\ 72Rr \leq \sum \frac{a^2}{\sin^2 \frac{A}{2}} \leq 36R^2.$$

Soluție

In ΔABC

$$\sum \frac{a^2}{\sin^2 \frac{A}{2}} = 8R(4R+r).$$

Inegalitatea din dreapta.

$$LHS = \sum \frac{a^2}{\sin^2 \frac{A}{2}} = 8R(4R+r) \overset{Euler}{\leq} 8R\left(4R+\frac{R}{2}\right) = 36R^2 = RHS .$$

Inegalitatea din stânga.

$$\sum \frac{a^2}{\sin^2 \frac{A}{2}} = 8R(4R+r) \overset{Euler}{\geq} 8R(4\cdot 2r+r) = 72Rr .$$

$$3)\sum \frac{a^2}{\sin^2 \frac{A}{2}} \leq 3\sum \frac{a^2}{\cos^2 \frac{A}{2}} .$$

Dezvoltări, Marin Chirciu

Soluție

Lema1.

In ΔABC

$$\sum \frac{a^2}{\sin^2 \frac{A}{2}} = 8R(4R+r).$$

Lema2.

In ΔABC

$$\sum \frac{a^2}{\cos^2 \frac{A}{2}} = 8R(2R-r).$$

Folosind **Lemele** inegalitatea se scrie:

$8R(4R+r) \leq 3\cdot 8R(2R-r) \Leftrightarrow (4R+r) \leq 3(2R-r) \Leftrightarrow R \geq 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema278.

Determinați numerele prime a,b,c ,care verifică egalitatea:

$$7a^2+35b^2-19c=2020.$$

OL,Mehedinți

Soluție

Se studiază după paritatea numerele prime a,b,c.

Dacă toate numerele prime a,b,c sunt impare $\Rightarrow$ contradicție.

Dacă toate numerele prime a,b,c sunt pare $\Rightarrow a=b=c=2 \Rightarrow$ contradicție.

Dacă două dintre numere prime a,b,c sunt pare $\Rightarrow$ membrul stâng=impar și membrul drept=2020=par $\Rightarrow$ contradicție.

Dacă $c=2 \Rightarrow 7a^2+35b^2-19\cdot 2=2020 \Rightarrow 7a^2+35b^2=2058 \Rightarrow a^2+5b^2=294 \Rightarrow$

$\Rightarrow b\in\{3,5,7\}$.

Pentru $b=3 \Rightarrow$ contradicție.

Pentru $b=5 \Rightarrow a=13$.

Pentru $b=7 \Rightarrow a=7$.

Numerele prime căutate sunt $(a,b,c)=(13,5,2),(7,7,2)$.

Remarca.

Determinați numerele prime a,b,c ,care verifică egalitatea:

$$5a^2+45b^2+387c=2024.$$

Marin Chirciu

Soluție

Se studiază după paritatea numerele prime a,b,c.

Dacă toate numerele prime a,b,c sunt impare $\Rightarrow$ contradicție.

Dacă toate numerele prime a,b,c sunt pare $\Rightarrow a=b=c=2 \Rightarrow$ contradicție.

Dacă două dintre numere prime a,b,c sunt pare $\Rightarrow$ membrul stâng=impar și membrul drept=2020=par $\Rightarrow$ contradicție.

Dacă $c = 2 \Rightarrow 5a^2 + 45b^2 + 387 \cdot 2 = 2024 \Rightarrow 5a^2 + 45b^2 = 1250 \Rightarrow a^2 + 9b^2 = 250 \Rightarrow$

$\Rightarrow b \in \{3,5\}$.

Pentru $b = 3 \Rightarrow$ contradicție.

Pentru $b = 5 \Rightarrow a = 5$.

Numerele prime căutate sunt $(a,b,c) = (5,5,2)$.

Problema279.

In ΔABC, M este mijlocul laturii BC și $AM \perp AC$. Atunci

$$BC^2 = AB^2 + 3AC^2.$$

Math Olymp 11/2024, Elton Papanikolla

Soluție

Lema.

In ΔABC, M este mijlocul laturii BC. Atunci

$$AB^2 + AC^2 = 2\left(AM^2 + MC^2\right).$$

Teorema lui Appolonius

Folosind Teorema lui Appolonius: $AB^2 + AC^2 = 2\left(AM^2 + MC^2\right)$ și teorema lui Pitagora în ΔMAC: $AM^2 + AC^2 = MC^2 \Rightarrow$

$$\Rightarrow AB^2 + AC^2 = 2AM^2 + 2MC^2 = 2\left(MC^2 - AC^2\right) + 2MC^2 = 4MC^2 - 2AC^2 = BC^2 - 2AC^2.$$

Din $AB^2 + AC^2 = BC^2 - 2AC^2 \Rightarrow BC^2 = AB^2 + 3AC^2$.

Problema280.

Solve in real numbers the equation

$$(x+1)\sqrt{x^2 + 2x + 2} + x\sqrt{x^2 + 1} = 0.$$

Bulgarian Mathematical Competition-2023, Nedyalka Dimitrova

Soluție

Lema.

Notând $\sqrt{x^2 + 2x + 2} = a > 0$ și $\sqrt{x^2 + 1} = b > 0 \Rightarrow$

$$x=\frac{\left(x^2+2x+2\right)-\left(x^2+1\right)-1}{2}=\frac{a^2-b^2-1}{2}\Rightarrow x+1=\frac{a^2-b^2+1}{2}.$$

Ecuația se scrie:

$$\frac{a^2-b^2+1}{2}\cdot a+\frac{a^2-b^2-1}{2}\cdot b=0\Leftrightarrow\left(a^2-b^2+1\right)\cdot a+\left(a^2-b^2-1\right)\cdot b=0\Leftrightarrow$$

$$\Leftrightarrow\left(a^2-b^2\right)\cdot a+\left(a^2-b^2\right)\cdot b+a-b=0\Leftrightarrow(a-b)\left[(a+b)^2+1\right]=0\Leftrightarrow a=b\Leftrightarrow$$

$$\Leftrightarrow\sqrt{x^2+2x+2}=\sqrt{x^2+1}\Leftrightarrow x^2+2x+2=x^2+1\Leftrightarrow 2x=-1\Leftrightarrow x=\frac{-1}{2}.$$

Deducem că ecuația admite soluția unică $x=\frac{-1}{2}$.

Remarca.

Let $\lambda\geq 0$ fixed.Solve in real numbers the equation

$$(x+\lambda)\sqrt{x^2+2x+2\lambda}+x\sqrt{x^2+\lambda}=0.$$

Marin Chirciu

Soluție

Lema.

Notând $\sqrt{x^2+2x+2\lambda}=a\geq 0$ și $\sqrt{x^2+\lambda}=b\geq 0\Rightarrow$

$$x=\frac{\left(x^2+2x+2\lambda\right)-\left(x^2+\lambda\right)-\lambda}{2}=\frac{a^2-b^2-\lambda}{2}\Rightarrow x+\lambda=\frac{a^2-b^2+\lambda}{2}.$$

Ecuația se scrie:

$$\frac{a^2-b^2+\lambda}{2}\cdot a+\frac{a^2-b^2-\lambda}{2}\cdot b=0\Leftrightarrow\left(a^2-b^2+\lambda\right)\cdot a+\left(a^2-b^2-\lambda\right)\cdot b=0\Leftrightarrow$$

$$\Leftrightarrow\left(a^2-b^2\right)\cdot a+\left(a^2-b^2\right)\cdot b+\lambda(a-b)=0\Leftrightarrow(a-b)\left[(a+b)^2+\lambda\right]=0\Leftrightarrow a=b\Leftrightarrow$$

$$\Leftrightarrow\sqrt{x^2+2x+\lambda}=\sqrt{x^2+\lambda}\Leftrightarrow x^2+2x+2\lambda=x^2+\lambda\Leftrightarrow 2x=-\lambda\Leftrightarrow x=\frac{-\lambda}{2}.$$

Deducem că ecuația admite soluția unică $x=\frac{-\lambda}{2}$.

Remarca.

Solve in real numbers the equation:

$$1)\ (x+2)\sqrt{x^2+2x+4}+x\sqrt{x^2+2}=0.$$

Ecuația admite soluția unică $x=-1$.

$$2)\ (x+3)\sqrt{x^2+2x+6}+x\sqrt{x^2+3}=0.$$

Ecuația admite soluția unică $x=\frac{-3}{2}$.

$$3)\ (x+5)\sqrt{x^2+2x+10}+x\sqrt{x^2+5}=0.$$

Ecuația admite soluția unică $x=\frac{-5}{2}$.

Problema281.

If $x, y, z>0$, $x+y+z=3$ then

$$\sum\frac{x}{\sqrt{2\left(y^4+z^4\right)+7yz}}\geq\frac{1}{3}.$$

Mathematical Inequalities 11/2024

Soluție

Lema.

If $x, y, z>0$ then

$$\frac{1}{\sqrt{2\left(y^4+z^4\right)+7yz}}\geq\frac{1}{2y^2+5yz+2z^2}.$$

Demonstrație

$$\frac{1}{\sqrt{2\left(y^4+z^4\right)+7yz}}\geq\frac{1}{2y^2+5yz+2z^2}\Leftrightarrow 2y^2-2yz+2z^2\geq\sqrt{2\left(y^4+z^4\right)}\Leftrightarrow$$

$$\Leftrightarrow y^4-4y^3z+6y^2z^2-4yz^3+z^4\geq 0\Leftrightarrow (y-z)^4\geq 0.$$

$$LHS=\sum\frac{x}{\sqrt{2\left(y^4+z^4\right)}+7yz}\overset{Lema}{\geq}\sum\frac{x}{2y^2+5yz+2z^2}=\sum\frac{x^2}{2xy^2+5xyz+2xz^2}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum\left(2xy^2+5xyz+2xz^2\right)}=$$

$$=\frac{\left(\sum x\right)^2}{2\sum x\left(y^2+z^2\right)+15xyz}=\frac{9}{2\left(9-\frac{1}{3}\sum x^3-2xyz\right)+15xyz}=\frac{27}{54-2\sum x^3+33xyz}\overset{(1)}{\geq}\frac{1}{3}=RHS,$$

unde $\frac{27}{54-2\sum x^3+33xyz}\overset{(1)}{\geq}\frac{1}{3}\Leftrightarrow 81\geq 54-2\sum x^3+33xyz\Leftrightarrow 2\sum x^3+27\geq 33xyz$,

veziAM-GM $\sum x^3\geq 3xyz$.

Rămâne să arătăm că:

$2\cdot 3xyz+27\geq 33xyz\Leftrightarrow xyz\leq 1$, vezi $3=x+y+z\geq 3\sqrt[3]{xyz}\Rightarrow xyz\leq 1$.

Am folosit mai sus:

$$\left(x+y+z\right)^3=\sum x^3+3\prod\left(x+y\right),\ x+y+z=3\Rightarrow 3^3=\sum x^3+6xyz+3\sum x\left(y^2+z^2\right)\Rightarrow$$

$$\Rightarrow\sum x\left(y^2+z^2\right)=9-\frac{1}{3}\sum x^3-2xyz.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Remarca.

If $x,y,z>0$, $x+y+z=3$ and $\lambda\geq 4$ then

$$\sum\frac{x}{\sqrt{2\left(y^4+z^4\right)}+\lambda yz}\geq\frac{3}{\lambda+2}.$$

Marin Chirciu

Soluție

Lema.

If $x,y,z>0$ and $\lambda\geq 4$ then

$$\frac{1}{\sqrt{2\left(y^4+z^4\right)}+\lambda yz}\geq\frac{1}{2y^2+\left(\lambda-2\right)yz+2z^2}.$$

Demonstrație

$$\frac{1}{\sqrt{2\left(y^4+z^4\right)}+\lambda yz} \geq \frac{1}{2y^2+(\lambda-2)yz+2z^2} \Leftrightarrow 2y^2-2yz+2z^2 \geq \sqrt{2\left(y^4+z^4\right)} \Leftrightarrow$$

$$\Leftrightarrow y^4-4y^3z+6y^2z^2-4yz^3+z^4 \geq 0 \Leftrightarrow (y-z)^4 \geq 0.$$

$$LHS=\sum \frac{x}{\sqrt{2\left(y^4+z^4\right)}+\lambda yz} \overset{Lema}{\geq} \sum \frac{x}{2y^2+(\lambda-2)yz+2z^2}=\sum \frac{x^2}{2xy^2+(\lambda-2)xyz+2xz^2} \overset{CS}{\geq}$$

$$\overset{CS}{\geq} \frac{\left(\sum x\right)^2}{\sum\left(2xy^2+(\lambda-2)xyz+2xz^2\right)}=\frac{\left(\sum x\right)^2}{2\sum x\left(y^2+z^2\right)+3(\lambda-2)xyz}=$$

$$\frac{9}{2\left(9-\frac{1}{3}\sum x^3-2xyz\right)+3(\lambda-2)xyz}=\frac{27}{54-2\sum x^3+(9\lambda-30)xyz} \overset{(1)}{\geq} \frac{3}{\lambda+2}=RHS,$$

unde $\frac{27}{54-2\sum x^3+(9\lambda-30)xyz} \overset{(1)}{\geq} \frac{3}{\lambda+2} \Leftrightarrow 9(\lambda+2) \geq 54-2\sum x^3+(9\lambda-30)xyz \Leftrightarrow$
$2\sum x^3+9(\lambda-4) \geq 3(3\lambda-10)xyz$, care rezultă din AM-GM $\sum x^3 \geq 3xyz$.

Rămâne să arătăm că:

$2\cdot 3xyz+9(\lambda-4) \geq 3(3\lambda-10)xyz \Leftrightarrow 9(\lambda-4)xyz \leq 9(\lambda-4) \overset{\lambda\geq 4}{\Leftrightarrow} xyz \leq 1$, vezi
$3=x+y+z \geq 3\sqrt[3]{xyz} \Rightarrow xyz \leq 1$.

Am folosit mai sus:

$$(x+y+z)^3=\sum x^3+3\prod(x+y),\ x+y+z=3 \Rightarrow 3^3=\sum x^3+6xyz+3\sum x\left(y^2+z^2\right) \Rightarrow$$

$$\Rightarrow \sum x\left(y^2+z^2\right)=9-\frac{1}{3}\sum x^3-2xyz.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Remarca.

CIf $x, y, z>0$, $x+y+z=3$ then

$$\sum \frac{x}{\sqrt{2\left(y^4+z^4\right)}+4yz} \geq \frac{1}{2}.$$

Soluție

Cazul $\lambda=4$ în inegalitatea de mai sus.

Remarca.

If $x, y, z > 0$, $x^2 + y^2 + z^2 = 3$ and $\lambda \ge 2$ then

$$\sum \frac{1}{\sqrt{2\left(y^4 + z^4\right) + \lambda yz}} \ge \frac{3}{\lambda + 2}.$$

Marin Chirciu

Soluție

Lema.

If $x, y, z > 0$ and $\lambda \ge 2$ then

$$\frac{1}{\sqrt{2\left(y^4 + z^4\right) + \lambda yz}} \ge \frac{1}{2y^2 + (\lambda - 2) yz + 2z^2}.$$

Demonstrație

$$\frac{1}{\sqrt{2\left(y^4 + z^4\right) + \lambda yz}} \ge \frac{1}{2y^2 + (\lambda - 2) yz + 2z^2} \Leftrightarrow 2y^2 - 2yz + 2z^2 \ge \sqrt{2\left(y^4 + z^4\right)} \Leftrightarrow$$

$$\Leftrightarrow y^4 - 4y^3 z + 6y^2 z^2 - 4yz^3 + z^4 \ge 0 \Leftrightarrow (y - z)^4 \ge 0.$$

$$\sum \frac{1}{\sqrt{2\left(y^4 + z^4\right) + \lambda yz}} \overset{Lema}{\ge} \sum \frac{1}{2y^2 + (\lambda - 2) yz + 2z^2} \overset{CS}{\ge}$$

$$\overset{CS}{\ge} \frac{9}{\sum\left(2y^2 + (\lambda - 2) yz + 2z^2\right)} = \frac{9}{2\sum\left(y^2 + z^2\right) + (\lambda - 2)\sum yz} = \frac{9}{4\sum x^2 + (\lambda - 2)\sum yz} \overset{SOS}{\ge}$$

$$= \frac{9}{4\sum x^2 + (\lambda - 2)\sum x^2} \overset{SOS}{\ge} \frac{9}{(\lambda + 2)\sum x^2} = \frac{9}{(\lambda + 2)\cdot 3} = \frac{3}{\lambda + 2} = RHS.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

If $x, y, z > 0$, $x^2 + y^2 + z^2 = 3$ then

$$\sum \frac{1}{\sqrt{2\left(y^4 + z^4\right) + 2yz}} \ge \frac{3}{4}.$$

Soluție

Cazul $\lambda = 2$ în inegalitatea de mai sus.

Problema282.

If $a,b,c>0$, $a+b+c=3$ then

$$\sum\sqrt{\frac{ab}{2(b+1)}}\le\frac{3}{2}.$$

RMM 11/2024, Konstantinos Geronikolas, Greece

Soluție

$$LHS=\sum\sqrt{\frac{ab}{2(b+1)}}\overset{AM-GM}{\le}\sum\sqrt{\frac{ab}{4\sqrt{b}}}=\frac{1}{2}\sum\sqrt{a}\sqrt[4]{b}\overset{CBS}{\le}\frac{1}{2}\sqrt{\sum a\sum\sqrt{a}}\overset{CBS}{\le}\frac{1}{2}\sqrt{3\cdot 3}=\frac{3}{2}=RHS.$$

Am folosit mai sus: $\sum\sqrt{a}\overset{CBS}{\le}\sqrt{3\sum a}=\sqrt{3\cdot 3}=3$.

Remarca.

If $a,b,c,\lambda>0$, $a+b+c=3\lambda$ then

$$\sum\sqrt{\frac{ab}{2(b+\lambda)}}\le\frac{3}{2}\sqrt{\lambda}.$$

Marin Chirciu

Soluție

$$LHS=\sum\sqrt{\frac{ab}{2(b+\lambda)}}\overset{AM-GM}{\le}\sum\sqrt{\frac{ab}{4\sqrt{\lambda b}}}=\frac{1}{2\sqrt[4]{\lambda}}\sum\sqrt{a}\sqrt[4]{b}\overset{CBS}{\le}\frac{1}{2\sqrt[4]{\lambda}}\sqrt{\sum a\sum\sqrt{b}}\overset{CBS}{\le}$$

$$\overset{SOS}{\le}\frac{1}{2\sqrt[4]{\lambda}}\sqrt{3\lambda\cdot 3\sqrt{\lambda}}=\frac{3}{2\sqrt[4]{\lambda}}\sqrt{\lambda}\sqrt[4]{\lambda}=\frac{3}{2}\sqrt{\lambda}=RHS.$$

Am folosit mai sus: $\sum\sqrt{a}\overset{CBS}{\le}\sqrt{3\sum a}=\sqrt{3\cdot 3\lambda}=3\sqrt{\lambda}$.

Problema283.

If $a,b,c>0$, $a^2+b^2+c^2=3abc$ then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3}{a+b+c}\ge 4.$$

Mathematical Inequalities 11/2024, Witcher Ben

Soluție

Folosim pqr -Method.

Notăm $p=a+b+c, q=ab+bc+ca, r=abc$.

Lema.

If $a,b,c>0$ then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3\sqrt[3]{\frac{a+b+c}{3}}.$$

Demonstrație

$$\left(\sum\frac{a}{b}\right)^2=\sum\frac{a^2}{b^2}+2\sum\frac{a}{c}=\sum\left(\frac{a^2}{b^2}+2\frac{a}{c}\right)=\sum\left(\frac{a^2}{b^2}+\frac{a}{c}+\frac{a}{c}\right)=\sum\left(\frac{a^2}{b^2}+\frac{a^2}{ac}+\frac{a^2}{ac}\right)=$$

$$=\sum a^2\left(\frac{1}{b^2}+\frac{1}{ac}+\frac{1}{ac}\right)\ge\sum a^2\cdot 3\sqrt[3]{\frac{1}{b^2}\cdot\frac{1}{ac}\cdot\frac{1}{ac}}=\sum a^2\cdot 3\sqrt[3]{\frac{1}{a^2b^2c^2}}=\sum a^2\cdot 3\sqrt[3]{\frac{1}{r^2}}=\sum\frac{3a^2}{\sqrt[3]{r^2}}=$$

$$=\sum\frac{3a^2}{\sqrt[3]{r^2}}=\frac{3\sum a^2}{\sqrt[3]{r^2}}=\frac{3\cdot 3r}{\sqrt[3]{r^2}}=\frac{9r}{\sqrt[3]{r^2}}=9\sqrt[3]{r}=9\sqrt[3]{\frac{\sum a^2}{3}}\overset{CS}{\ge}9\sqrt[3]{\frac{\frac{1}{3}\left(\sum a\right)^2}{3}}=9\sqrt[3]{\left(\frac{\sum a}{3}\right)^2}.$$

Din $\left(\sum\frac{a}{b}\right)^2\ge 9\sqrt[3]{\left(\frac{\sum a}{3}\right)^2}\Rightarrow\sum\frac{a}{b}\ge 3\sqrt[3]{\frac{\sum a}{3}}$.

$$LHS=\sum\frac{a}{b}+\frac{3}{\sum a}\overset{Lema}{\ge}3\sqrt[3]{\frac{\sum a}{3}}+\frac{3}{\sum a}=3\sqrt[3]{\frac{p}{3}}+\frac{3}{p}=\sqrt[3]{\frac{p}{3}}+\sqrt[3]{\frac{p}{3}}+\sqrt[3]{\frac{p}{3}}+\frac{3}{p}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}4\sqrt[4]{\sqrt[3]{\frac{p}{3}}\cdot\sqrt[3]{\frac{p}{3}}\cdot\sqrt[3]{\frac{p}{3}}\cdot\frac{3}{p}}=4\sqrt[4]{\frac{p}{3}\cdot\frac{3}{p}}=4=RHS.$$

Egalitatea în AM-GM are loc pentru $\sqrt[3]{\frac{p}{3}}=\frac{3}{p}\Leftrightarrow p^4=81\Leftrightarrow p=3$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $a^2+b^2+c^2=3abc$ and $n\in\mathbf{N}$ then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3^{n+1}}{a+b+c}\ge(n+4)\sqrt[n+4]{\left(\frac{3}{a+b+c}\right)^n}.$$

Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm $p = a+b+c, q = ab+bc+ca, r = abc$.

Lema.

If $a,b,c>0$ then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3\sqrt[3]{\frac{a+b+c}{3}}.$$

$$LHS=\sum\frac{a}{b}+\frac{3^{n+1}}{\sum a}\overset{Lema}{\ge}3\sqrt[3]{\frac{\sum a}{3}}+\frac{3^{n+1}}{\sum a}=3\sqrt[3]{\frac{p}{3}}+\frac{3^{n+1}}{p}=\sqrt[3]{\frac{p}{3}}+\sqrt[3]{\frac{p}{3}}+\sqrt[3]{\frac{p}{3}}+\frac{3^{n+1}}{p}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}(n+4)\sqrt[n+4]{\sqrt[3]{\frac{p}{3}}\cdot\sqrt[3]{\frac{p}{3}}\cdot\sqrt[3]{\frac{p}{3}}\cdot\underbrace{\frac{3}{p}\cdot\ldots\cdot\frac{3}{p}}_{n+1}}=(n+4)\sqrt[n+4]{\frac{p}{3}\cdot\left(\frac{3}{p}\right)^{n+1}}=(n+4)\sqrt[n+4]{\left(\frac{3}{p}\right)^{n}}=RHS.$$

Egalitatea în AM-GM are loc pentru $\sqrt[3]{\frac{p}{3}}=\frac{3}{p}\Leftrightarrow p^4=81\Leftrightarrow p=3$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $a^2+b^2+c^2=3abc$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^3\ge 9(a+b+c).$$

Marin Chirciu

Problema284.

If $a,b,c>0$, $a^3+b^3+c^3=3$ then

$$\frac{18}{\sum a\sum ab}\ge 1+\frac{\sum a^2}{\sum ab}.$$

Mathematical Inequalities 11/2024, Nguyen Minh Tho, Vietnam

Soluție

$$LHS=\frac{18}{\sum a\sum ab}\overset{\sum a^3=3}{=}\frac{6\sum a^3}{\sum a\sum ab}\overset{Chebyshev}{\ge}\frac{6\cdot\frac{1}{3}\sum a^2\sum a}{\sum a\sum ab}=\frac{2\sum a^2}{\sum ab}\overset{SOS}{\ge}1+\frac{\sum a^2}{\sum ab}=RHS.$$

Am folosit mai sus inegalitatea lui Chebyshev $\sum a^3\ge\frac{1}{3}\sum a^2\sum a$.

(SOS) $\Leftrightarrow \frac{2\sum a^2}{\sum ab} \overset{SOS}{\geq} 1+\frac{\sum a^2}{\sum ab} \Leftrightarrow \sum a^2 \geq \sum ab \Leftrightarrow \sum(a-b)^2 \geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $a^3+b^3+c^3=3$ and $\lambda \geq 0$ then

$$\frac{9\lambda}{\sum a\sum ab} \geq 1+(\lambda-1)\frac{\sum a^2}{\sum ab}.$$

Marin Chirciu

Soluție

$$LHS=\frac{9\lambda}{\sum a\sum ab} \overset{\sum a^3=3}{=} \frac{3\lambda\sum a^3}{\sum a\sum ab} \overset{Chebyshev}{\geq} \frac{3\lambda\cdot\frac{1}{3}\sum a^2\sum a}{\sum a\sum ab} = \frac{\lambda\sum a^2}{\sum ab} \overset{SOS}{\geq} 1+(\lambda-1)\frac{\sum a^2}{\sum ab}=RHS.$$

(SOS) $\Leftrightarrow \frac{\lambda\sum a^2}{\sum ab} \overset{SOS}{\geq} 1+(\lambda-1)\frac{\sum a^2}{\sum ab} \Leftrightarrow \sum a^2 \geq \sum ab \Leftrightarrow \sum(a-b)^2 \geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema285.

If $x,y,z>0$,solve the system:

$$\begin{cases} x^2+y^2+z^2=3 \\ \frac{x^2-3}{x^2-4}+\frac{y^2-3}{y^2-4}+\frac{z^2-3}{z^2-4}=2 \end{cases}.$$

MathAtelier 11/2024,Panagiotis Danousis, Greece

Soluție

Cu substituția $(x^2,y^2,z^2)=(a,b,c)$ sistemul devine $\begin{cases} a+b+c=3 \\ \frac{1}{4-a}+\frac{1}{4-b}+\frac{1}{4-c}=1 \end{cases}$.

Folosind inegalitatea lui Jensen pentru funcția convexă $f(t)=\frac{1}{4-t}$, $t\in(0,4)$ obținem

$\sum\frac{1}{4-a}=\sum f(a) \geq 3f\left(\frac{a+b+c}{3}\right)=3f(1)=3\cdot\frac{1}{3}=1$, cu egalitate pentru $a=b=c=1$.

Sistemul inițial admite soluția unică $(x,y,z)=(1,1,1)$.

Remarca.

1). Let $\lambda>0$ fixed. If $x,y,z>0$,solve the system:

$$\begin{cases} x^3+y^3+z^3=3 \\ \dfrac{x^3-\lambda}{x^3-\lambda-1}+\dfrac{y^3-\lambda}{y^3-\lambda-1}+\dfrac{z^3-\lambda}{z^3-\lambda-1}=3\left(1-\dfrac{1}{\lambda}\right) \end{cases}.$$

Soluție

Cu substituția $(x^3,y^3,z^3)=(a,b,c)$ sistemul devine $\begin{cases} a+b+c=3 \\ \dfrac{1}{\lambda+1-a}+\dfrac{1}{\lambda+1-b}+\dfrac{1}{\lambda+1-c}=\dfrac{3}{\lambda} \end{cases}.$

Folosind inegalitatea lui Jensen pentru funcția convexă $f(t)=\dfrac{1}{\lambda+1-t}, t\in(0,\lambda+1)$ obținem

$\sum\dfrac{1}{\lambda+1-a}=\sum f(a)\geq 3f\left(\dfrac{a+b+c}{3}\right)=3f(1)=3\cdot\dfrac{1}{\lambda}=\dfrac{3}{\lambda}$, cu egalitate pentru $a=b=c=1$.

Sistemul inițial admite soluția unică $(x,y,z)=(1,1,1)$.

Remarca.

2). Let $\lambda>0$ fixed. If $x,y,z,t>0$,solve the system:

$$\begin{cases} x^3+y^3+z^3+t^3=4 \\ \dfrac{x^3-\lambda}{x^3-\lambda-1}+\dfrac{y^3-\lambda}{y^3-\lambda-1}+\dfrac{z^3-\lambda}{z^3-\lambda-1}+\dfrac{t^3-\lambda}{t^3-\lambda-1}=4\left(1-\dfrac{1}{\lambda}\right) \end{cases}.$$

Soluție

Cu substituția $(x^3,y^3,z^3,t^3)=(a,b,c,d)$ sistemul devine

$$\begin{cases} a+b+c+d=4 \\ \dfrac{1}{\lambda+1-a}+\dfrac{1}{\lambda+1-b}+\dfrac{1}{\lambda+1-c}+\dfrac{1}{\lambda+1-d}=\dfrac{4}{\lambda} \end{cases}.$$

Folosind inegalitatea lui Jensen pentru funcția convexă $f(t)=\dfrac{1}{\lambda+1-t}, t\in(0,\lambda+1)$ obținem

$$\sum\dfrac{1}{\lambda+1-a}=\sum f(a)\geq 4f\left(\dfrac{a+b+c+d}{4}\right)=4f(1)=4\cdot\dfrac{1}{\lambda}=\dfrac{4}{\lambda},$$

cu egalitate pentru $a=b=c=d=1$.

Sistemul inițial admite soluția unică $(x,y,z,t)=(1,1,1,1)$.

Remarca.

3). Let $\lambda>0$ fixed. If $x,y>0$,solve the system:

$$\begin{cases} x^2+y^2=2 \\ \dfrac{x^2-\lambda}{x^2-\lambda-1}+\dfrac{y^2-\lambda}{y^2-\lambda-1}=2\left(1-\dfrac{1}{\lambda}\right) \end{cases}.$$

Dezvoltări, Marin Chirciu

Soluție

Cu substituția $(x^2,y^2)=(a,b)$ sistemul devine $\begin{cases} a+b=2 \\ \dfrac{1}{\lambda+1-a}+\dfrac{1}{\lambda+1-b}=\dfrac{2}{\lambda} \end{cases}$.

Folosind inegalitatea lui Jensen pentru funcția convexă $f(t)=\dfrac{1}{\lambda+1-t}, t\in(0,\lambda+1)$ obținem

$\sum\dfrac{1}{\lambda+1-a}=\sum f(a)\geq 2f\left(\dfrac{a+b}{2}\right)=2f(1)=2\cdot\dfrac{1}{\lambda}=\dfrac{2}{\lambda}$, cu egalitate pentru $a=b=1$.

Sistemul inițial admite soluția unică $(x,y)=(1,1)$.

Problema286.

In ΔABC

$$\sum\cot A+\prod\cot\frac{A}{2}\geq 2\sum\csc A.$$

RMM 11/2024,Nguyen Hung Cuong, Vietnam

Soluție

Folosind identitățile în triunghi:

$\sum\cot A=\dfrac{p^2-r^2-4Rr}{2pr}, \prod\cot\dfrac{A}{2}=\dfrac{p}{r}$ și $\sum\csc A=\dfrac{p^2+r^2+4Rr}{2pr}$, inegalitatea se scrie:

$\dfrac{p^2-r^2-4Rr}{2pr}+\dfrac{p}{r}\geq\dfrac{p^2+r^2+4Rr}{2pr}\Leftrightarrow p^2\geq 12Rr+3r^2$, vezi Gerretsen $p^2\geq 16Rr-5r^2$.

Remarca.

In ΔABC

$$1)\sum\cot A+\lambda\prod\cot\frac{A}{2}\ge\frac{1}{2}(3\lambda+1)\sum\csc A,\lambda\ge 0.$$

Marin Chirciu

Soluție

Folosind identitățile în triunghi:

$\sum\cot A=\frac{p^2-r^2-4Rr}{2pr}$, $\prod\cot\frac{A}{2}=\frac{p}{r}$ și $\sum\csc A=\frac{p^2+r^2+4Rr}{2pr}$, inegalitatea se scrie:

$\frac{p^2-r^2-4Rr}{2pr}+\lambda\frac{p}{r}\ge\frac{1}{2}(3\lambda+1)\frac{p^2+r^2+4Rr}{2pr}\Leftrightarrow p^2\ge 12Rr+3r^2$, vezi $p^2\ge 16Rr-5r^2$.

In acute ΔABC

$$2)\sum\tan A+\sum\sec A\ge 9\left(3+2\sqrt{3}\right)\prod\tan\frac{A}{2}.$$

Marin Chirciu

Soluție

Folosind relațiile în triunghi:

$\sum\tan A=\frac{2pr}{p^2-(2R+r)^2}\ge 3\sqrt{3}$, $\sum\sec A=\frac{p^2+r^2-4R^2}{p^2-(2R+r)^2}\ge 6$ și $\prod\tan\frac{A}{2}=\frac{r}{p}$.

$LHS=\sum\tan A+\sum\sec A\ge 3\sqrt{3}+6\overset{(1)}{\ge}9\left(3+2\sqrt{3}\right)\prod\tan\frac{A}{2}=RHS$,

unde $3\sqrt{3}+6\overset{(1)}{\ge}9\left(3+2\sqrt{3}\right)\prod\tan\frac{A}{2}\Leftrightarrow 3\sqrt{3}+6\ge 9\left(3+2\sqrt{3}\right)\frac{r}{p}\Leftrightarrow\frac{p}{r}\ge 3\sqrt{3}$,vezi $p\ge 3r\sqrt{3}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema287.

S682. If $a,b,c>0$ then

$$\frac{a^2}{2bc}+\frac{b}{a+b}+\frac{c}{a+c}\ge\frac{a}{b+c}+1.$$

Mathematical Reflections 6/2024, Nguyen Viet Hung, Vietnam

Solution.

$$LHS=\frac{a^2}{2bc}+\frac{b^2}{ab+b^2}+\frac{c^2}{ac+c^2}\overset{Titu-Lemma}{\geq}\frac{(a+b+c)^2}{b^2+c^2+2bc+ab+ac}=\frac{(a+b+c)^2}{(b+c)^2+a(b+c)}=$$

$$=\frac{(a+b+c)^2}{(b+c)(a+b+c)}=\frac{a+b+c}{b+c}=\frac{a}{b+c}+1=RHS.$$

Equality occurs if and only if $a=b=c$.

Remarca.

If $a,b,c>0$ then

$$\frac{a^3}{2bc}+\frac{b^2}{a+b}+\frac{c^2}{a+c}\geq a+\frac{bc}{b+c}.$$

Marin Chirciu

Solution.

$$LHS=\frac{a^3}{2bc}+\frac{b^2}{a+b}+\frac{c^2}{a+c}=\frac{a^3}{2bc}+\frac{b^3}{ab+b^2}+\frac{c^3}{ac+c^2}\overset{Holder}{\geq}\frac{(a+b+c)^3}{3(b^2+c^2+2bc+ab+ac)}=$$

$$=\frac{(a+b+c)^3}{3\left[(b+c)^2+a(b+c)\right]}=\frac{(a+b+c)^3}{3(b+c)(a+b+c)}=\frac{(a+b+c)^2}{3(b+c)}\overset{SOS}{\geq}\frac{3(ab+bc+ca)}{3(b+c)}=$$

$$=a+\frac{bc}{b+c}=RHS.$$

Equality occurs if and only if $a=b=c$.

Remarca.

If $a,b,c>0$ and $n\in\mathbf{N}^*$ then

$$\frac{a^{n+1}}{2bc}+\frac{b^n}{a+b}+\frac{c^n}{a+c}\geq\frac{3}{b+c}\left(\frac{a+b+c}{3}\right)^n.$$

Marin Chirciu

Solution.

$$LHS=\frac{a^{n+1}}{2bc}+\frac{b^n}{a+b}+\frac{c^n}{a+c}=\frac{a^{n+1}}{2bc}+\frac{b^{n+1}}{ab+b^2}+\frac{c^{n+1}}{ac+c^2}\overset{Holder}{\geq}\frac{(a+b+c)^{n+1}}{3^{n-1}(b^2+c^2+2bc+ab+ac)}=$$

$$=\frac{(a+b+c)^{n+1}}{3^{n-1}\left[(b+c)^2+a(b+c)\right]}=\frac{(a+b+c)^{n+1}}{3^{n-1}(b+c)(a+b+c)}=\frac{(a+b+c)^n}{3^{n-1}(b+c)}=\frac{3}{b+c}\left(\frac{a+b+c}{3}\right)^n=$$

$= RHS$.

Equality occurs if and only if $a=b=c$.

Problema287.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{3}{x+y+z}\geq 4.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{x}{y}+\frac{3}{\sum x}=\sum\frac{x^2}{xy}+\frac{3}{\sum x}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum xy}+\frac{3}{\sum x}=\frac{p^2}{3}+\frac{3}{p}\overset{(1)}{\geq}4=RHS,$$

unde $\frac{p^2}{3}+\frac{3}{p}\overset{(1)}{\geq}4\Leftrightarrow p^3-12p+9\geq 0$, vezi $(p-3)\left(p^2+3p-3\right)\geq 0$, vezi $p\geq 3$, care rezultă din $p^2=(x+y+z)^2\geq 3(xy+yz+zx)=3\cdot 3=9\Rightarrow p^2\geq 9\Rightarrow p\geq 3$.

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Remarca.

If $x,y,z>0$, $xy+yz+zx=3$ and $\lambda\leq 18$ then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{\lambda}{x+y+z}\geq 3+\frac{\lambda}{3}.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{x}{y}+\frac{\lambda}{\sum x}=\sum\frac{x^2}{xy}+\frac{\lambda}{\sum x}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum xy}+\frac{\lambda}{\sum x}=\frac{p^2}{3}+\frac{\lambda}{p}\overset{(1)}{\geq}3+\frac{\lambda}{3}=RHS,$$

unde $\frac{p^2}{3}+\frac{\lambda}{p}\overset{(1)}{\geq}3+\frac{\lambda}{3}\Leftrightarrow p^3-(\lambda+9)p+3\lambda\geq 0$, vezi $(p-3)\left(p^2+3p-\lambda\right)\geq 0$, vezi $p\geq 3$, care rezultă din $p^2=(x+y+z)^2\geq 3(xy+yz+zx)=3\cdot 3=9\Rightarrow p^2\geq 9\Rightarrow p\geq 3$.

Pentru $p \geq 3 \Rightarrow (p-3)(p^2+3p-\lambda) \geq 0$, $\lambda \leq 18$.

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{18}{x+y+z} \geq 9.$$

Soluție

Cazul $\lambda = 18$ mai sus.

Remarca.

In ΔABC

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}+\frac{p\sqrt{3}}{4R+r} \geq 4.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{3}{x+y+z} \geq 4.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\frac{\sqrt{3}\tan\frac{A}{2}}{\sqrt{3}\tan\frac{B}{2}}+\frac{\sqrt{3}\tan\frac{B}{2}}{\sqrt{3}\tan\frac{C}{2}}+\frac{\sqrt{3}\tan\frac{C}{2}}{\sqrt{3}\tan\frac{A}{2}}+\frac{3}{\sqrt{3}\tan\frac{A}{2}+\sqrt{3}\tan\frac{B}{2}+\sqrt{3}\tan\frac{C}{2}} \geq 4 \Leftrightarrow$$

$$\Leftrightarrow \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}+\frac{p\sqrt{3}}{4R+r}\geq 4.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema288.

Aflați numerele naturale $\overline{abcd}$ știind că $\left(\overline{ab}-\overline{cd}\right)^3=\overline{ad}$.

RMT, Mihai Vijdeluc, Baia Mare

Soluție

$\overline{ad}$ este cub perfect $\Rightarrow$ $\overline{ad}\in\{27,64\}$.

Cazul 1). $\overline{ad}=27\Rightarrow\overline{abcd}=2017$.

Cazul 1). $\overline{ad}=64\Rightarrow\overline{abcd}=6864$.

Problema289.

Determinați cifrele x,y,a,b,c , știind că $\left(\overline{xy}\right)^4=\overline{abc321}$

RMT, Aurel Doboșan, Lugoj

Soluție

$\left(\overline{xy}\right)^4$ trebuie să fie un număr de 6 cifre $\Rightarrow 18\leq\overline{xy}\leq 31$.

Avem $19^4=130321$.

Deducem $(x,y,a,b,c)=(1,9,1,3,0)$.

Problema290.

Aflați numerele prime p și q pentru care $p=2017-(3q)^3$.

RMT, Mihai Vijdeluc, Baia Mare

Soluție

$p=2017-27q^3\Rightarrow 27q^3=2017-p<2017\Rightarrow q\leq 4$, q =prim $\Rightarrow q\in\{2,3\}$.

Convine $q=2$ și $p=1801$.

Problema291.

Solve in real numbers the equation

$$\sqrt{x+3}+\sqrt{5-x}+4\sqrt{x}=2(x+3).$$

THCS 11/2024.

Soluție

Domeniul de definiție este $[0,5]$.

$$LHS=\sqrt{x+3}+\sqrt{5-x}+4\sqrt{x}=\frac{1}{2}\sqrt{4(x+3)}+\frac{1}{2}\sqrt{4(5-x)}+4\sqrt{x}\overset{AM-GM}{\le}$$

$\overset{AM-GM}{\le}\frac{1}{2}\cdot\frac{4+(x+3)}{2}+\frac{1}{2}\cdot\frac{4+(5-x)}{2}+4\cdot\frac{x+1}{2}=2(x+3)=RHS$, cu egalitate pentru

$4=(x+3),4=(5-x),x=1\Leftrightarrow x=1$.

Deducem că ecuația admite soluția unică $x=1$.

Remarca.

Let $\lambda>0$ fixed. Solve in real numbers the equation

$$\sqrt{x+\lambda^2-1}+\sqrt{\lambda^2+1-x}+4\lambda\sqrt{x}=2\lambda(x+2).$$

Marin Chirciu

Soluție

Domeniul de definiție este $\left[0,\lambda^2+1\right]$.

$$LHS=\sqrt{x+\lambda^2-1}+\sqrt{\lambda^2+1-x}+4\lambda\sqrt{x}\overset{CBS\&AM-GM}{\le}\sqrt{2\left(x+\lambda^2-1+\lambda^2+1-x\right)}+4\lambda\cdot\frac{x+1}{2}=$$

$=2\lambda+2\lambda(x+1)=2\lambda(x+2)=RHS$, cu egalitate pentru $x+\lambda^2-1=\lambda^2+1-x$, $x=1$.

Deducem că ecuația admite soluția unică $x=1$.

Problema292.

Deterrminați numerele naturale $\overline{ab}$ cu proprietatea că $\frac{\overline{ab}}{a+b}=a$.

RMT, Eugen Predoiu,Călărași

Soluție

Deducem că $\overline{ab} = 48$.

Problema293.

Deterrminați numerele naturale x, y, z care verifică simultan relațiile:

$xy + xz = 14, yz + yx = 18, zx + xy = 20$.

RMT, Mircea Mario Stoica,Arad

Soluție

Obținem $xy + xz + yz = 26 \Rightarrow xy = 6, xz = 8, yz = 12 \Rightarrow xyz = 24$

Deducem că $(x, y, z) = (2,3,4)$.

Problema294.

Fie a,b,c cifre nenule și $K = \dfrac{\overline{ab}+\overline{ba}}{a \cdot b}(a^2+b^2-c^2)$, $L = \dfrac{\overline{bc}+\overline{cb}}{b \cdot c}(b^2+c^2-a^2)$ și

$M = \dfrac{\overline{ca}+\overline{ac}}{c \cdot a}(c^2+a^2-b^2)$.

Arătați că suma $K + L + M \in \mathbf{N}$.

RMT, Radu Ghenghiu, Oradea

Soluție

Avem $K = \dfrac{11(a+b)}{ab}(a^2+b^2-c^2)$ și analoagele.

Obținem $K + L + M = \sum \dfrac{11(a+b)}{ab}(a^2+b^2-c^2) = 22(a+b+c) \in \mathbf{N}$

Problema295.

In ΔABC

$$\sum \frac{w_a^2}{ab} \geq \frac{9}{4}\sqrt{\frac{6R-3r}{5R-r}}.$$

RMM 11/2024, Dang Ngoc Minh, Vietnam

Soluție

Lema.

In ΔABC

$$\sum \frac{w_a^2}{ab} \geq \frac{27}{8p^2}\left(p^2 - r^2 - 4Rr\right).$$

Demonstrație

$$\sum \frac{w_a^2}{ab} = \sum \frac{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}{ab} = \sum \frac{\frac{4b^2c^2}{(b+c)^2}\cdot\frac{p(p-a)}{bc}}{ab} = \sum \frac{4p(p-a)c}{a(b+c)^2} =$$

$$= \sum \frac{p(p-a)c}{a\left(\frac{b+c}{2}\right)^2} \overset{AM-GM}{\geq} \sum \frac{p(p-a)c}{\left(\frac{a+\frac{b+c}{2}+\frac{b+c}{2}}{3}\right)^3} = \sum \frac{p(p-a)c}{\left(\frac{a+b+c}{3}\right)^3} = \sum \frac{p(p-a)c}{\left(\frac{2p}{3}\right)^3} =$$

$$= \sum \frac{27p(p-a)c}{8p^3} = \sum \frac{27(p-a)c}{8p^2} = \frac{27}{8p^2}\sum (p-a)c = \frac{27}{8p^2}\left(p\sum c - \sum ac\right) =$$

$$\frac{27}{8p^2}\left(p\cdot 2p - p^2 - r^2 - 4Rr\right) = \frac{27}{8p^2}\left(p^2 - r^2 - 4Rr\right).$$

$$LHS = \sum \frac{w_a^2}{ab} \overset{Lema}{\geq} \frac{27}{8p^2}\left(p^2 - r^2 - 4Rr\right) \overset{Mitrinovic\&Euler}{\geq} \frac{9}{4}\sqrt{\frac{6R-3r}{5R-r}} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\frac{9}{4} \leq \sum \frac{w_a^2}{ab} \leq \frac{9R}{8r}.$$

Marin Chirciu

Soluție

Inegalitatea din dreapta.

$$\sum \frac{w_a^2}{ab} = \sum \frac{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}{ab} = \sum \frac{\frac{4b^2c^2}{(b+c)^2}\cdot\frac{p(p-a)}{bc}}{ab} = \sum \frac{4p(p-a)c}{a(b+c)^2} \overset{AM-GM}{\leq} \sum \frac{4p(p-a)c}{a\cdot 4bc} =$$

$$\sum\frac{p(p-a)c}{a\cdot bc}=\frac{p}{abc}\sum(p-a)c=\frac{p}{4Rrp}\left(p\sum c-\sum ac\right)=\frac{1}{4Rr}\left(p\cdot 2p-p^2-r^2-4Rr\right)=$$

$$=\frac{1}{4Rr}\left(p^2-r^2-4Rr\right)\overset{Gerretsen}{\leq}\frac{1}{4Rr}\left(4R^2+4Rr+3r^2-r^2-4Rr\right)=\frac{4R^2+2r^2}{4Rr}=$$

$$=\frac{2R^2+r^2}{2Rr}\overset{Euler}{\leq}\frac{\frac{9R^2}{4}}{2Rr}=\frac{9R}{8r}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{w_a^2}{ab}=\sum\frac{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}{ab}=\sum\frac{\frac{4b^2c^2}{(b+c)^2}\cdot\frac{p(p-a)}{bc}}{ab}=\sum\frac{4p(p-a)c}{a(b+c)^2}=$$

$$=\sum\frac{p(p-a)c}{a\left(\frac{b+c}{2}\right)^2}\overset{AM-GM}{\geq}\sum\frac{p(p-a)c}{\left(\frac{a+\frac{b+c}{2}+\frac{b+c}{2}}{3}\right)^3}=\sum\frac{p(p-a)c}{\left(\frac{a+b+c}{3}\right)^3}=\sum\frac{p(p-a)c}{\left(\frac{2p}{3}\right)^3}=$$

$$=\sum\frac{27p(p-a)c}{8p^3}=\sum\frac{27(p-a)c}{8p^2}=\frac{27}{8p^2}\sum(p-a)c=\frac{27}{8p^2}\left(p\sum c-\sum ac\right)=$$

$$\frac{27}{8p^2}\left(p\cdot 2p-p^2-r^2-4Rr\right)=\frac{27}{8p^2}\left(p^2-r^2-4Rr\right)=\frac{27}{8}\left(1-\frac{4Rr+r^2}{p^2}\right)\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{27}{8}\left(1-\frac{4Rr+r^2}{16Rr-5r^2}\right)=\frac{27}{8}\left(1-\frac{4R+r}{16R-5r}\right)=\frac{27}{8}\left(\frac{12R-6r}{16R-5r}\right)\overset{Euler}{\geq}\frac{27}{8}\cdot\frac{2}{3}=\frac{9}{4}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema296.

If $a,b,c>0$ then

$$8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+15\geq 81\sum\frac{1}{a+b+5}.$$

Math Atelier 11/2024,Panagiotis Danousis, Greece

Soluție

Lema.

If $a,b,c>0$ then

$$\frac{81}{a+b+5}\le 5+\frac{4}{a}+\frac{4}{b}.$$

Demonstrație

$5+\frac{4}{a}+\frac{4}{b}=\frac{5^2}{5}+\frac{2^2}{a}+\frac{2^2}{b}\overset{CS}{\ge}\frac{(5+2+2)^2}{5+a+b+5}=\frac{81}{5+a+b}$,cu egalitate pentru $\frac{5}{5}=\frac{2}{a}=\frac{2}{b}\Leftrightarrow a=b=2$.

Obținem:

$$RHS=81\sum\frac{1}{a+b+5}\overset{Lema}{\le}\sum\left(5+\frac{4}{a}+\frac{4}{b}\right)=15+8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=LHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=2$.

Remarca.

If $a,b,c,\lambda,n>0$ then

$$3\lambda+2n^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge(\lambda+2n)^2\sum\frac{1}{a+b+\lambda}.$$

Marin Chirciu

Soluție

Lema.

If $a,b,\lambda,n>0$ then

$$\frac{(\lambda+2n)^2}{\lambda+a+b}\le\lambda+\frac{n^2}{a}+\frac{n^2}{b}.$$

Demonstrație

$\lambda+\frac{n^2}{a}+\frac{n^2}{b}=\frac{\lambda^2}{\lambda}+\frac{n^2}{a}+\frac{n^2}{b}\overset{CS}{\ge}\frac{(\lambda+n+n)^2}{\lambda+a+b}$, cu egalitate pentru $\frac{\lambda}{\lambda}=\frac{n}{a}=\frac{n}{b}$ $a=b=n$.

Obținem:

$$RHS=(\lambda+2n)^2\sum\frac{1}{a+b+\lambda}\overset{Lema}{\le}\sum\left(\lambda+\frac{n^2}{a}+\frac{n^2}{b}\right)=3\lambda+2n^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=LHS$$

Egalitatea are loc dacă și numai dacă $a=b=c=n$.

Problema297.

Determinați toate numerele de forma $\overline{abbc}$, cifrele a,b,c fiind distincte, știind că sunt îndeplinie simultan condițiile:

a) 13 divide $\overline{ba}$;

b) $\overline{ab}$ este număr prim;

c) c este pătrat perfect.

OJ-Teleorman, 1994

Soluție

Dina) și b) $\Rightarrow$ $\overline{ab} \in \{19,31\}$.

Dacă $\overline{ab} = 19 \Rightarrow \overline{abbc} \in \{1990,1994\}$.

Dacă $\overline{ab} = 31 \Rightarrow \overline{abbc} \in \{3110,,3114,3119\}$.

Problema are 5 soluții.

Problema298.

If $a,b,c>0$, $abc=1$

$$\sum a^3b\sum ab^3 \geq (a+b+c)^2.$$

IneMath 11/2024,Marin Chirciu

Soluție

$$LHS = \sum a^3b\sum ab^3 \overset{CBS}{\geq} \left(\sum a^2b^2\right)^2 \overset{(1)}{\geq} \left(\sum a\right)^2 = RHS,$$

unde $\left(\sum a^2b^2\right)^2 \overset{(1)}{\geq} \left(\sum a\right)^2 \Leftrightarrow \sum a^2b^2 \geq \sum a$, vezi $\sum a^2b^2 \geq abc\sum a = \sum a$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema299.

O682. If $a,b,c>0$ then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a+b+c}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}.$$

Mathematical Reflections 6/2024, Nguyen Viet Hung, Vietnam

Solution.

With substitution $(a,b,c)=(x^2,y^2,z^2)$ we have:

$$\frac{x^4+y^4+z^4}{x^2y^2+y^2z^2+z^2x^2}\geq\frac{x^2+y^2+z^2}{xy+yz+zx}\Leftrightarrow$$

$$\Leftrightarrow (x^4+y^4+z^4)(xy+yz+zx)\geq(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)\Leftrightarrow$$

$\Leftrightarrow \sum xy(x^4+y^4)+xyz\sum x^3\geq\sum x^2y^2(x^2+y^2)+3x^2y^2z^2$,which results from:

$\sum xy(x^4+y^4)\geq\sum x^2y^2(x^2+y^2)$, see $xy(x^4+y^4)\geq x^2y^2(x^2+y^2)\Leftrightarrow$

$\Leftrightarrow (x^4+y^4)\geq xy(x^2+y^2)\Leftrightarrow (x-y)(x^3-y^3)\geq 0\Leftrightarrow (x-y)^2(x^2+xy+y^2)\geq 0$,

and $xyz\sum x^3\geq 3x^2y^2z^2\Leftrightarrow\sum x^3\geq 3xyz$, see AM-GM.

Equality occurs if and only if $x=y=z\Leftrightarrow a=b=c$.

Remarca.

If $a,b,c>0$, $abc=1$ then

$$1)\ \frac{a^2+b^2+c^2}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\geq\frac{a+b+c}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}.$$

$$2)\ \frac{a^2+b^2+c^2}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\geq\frac{a+b+c}{ab+bc+ca}.$$

Dezvoltări, Marin Chirciu

Problema300.

Să se afle toate numerele prime $\overline{abc}$ astel încât $abc=567$.

Olimpiada Matematică din Republica Moldova, 1994

Soluție

Avem $567=3^4\cdot 7\Rightarrow\overline{abc}=979,997$.

Problema301.

În triunghiul ABC se înscrie rombul de Arie S, care are comun cu triunghiul unghiul A.

Să dse arate că

$$S\leq\frac{AB\cdot AC}{4}.$$

Constantin Apostol,Rm.Sărat, Concursul Laurențiu Duican, 1997

Soluție

Fie ADEF rombul, $D \in (AB), E \in (BC), F \in (AC)$.

Fie x =latura rombului.Din $\Delta EFC \sim \Delta BAC \Rightarrow \frac{x}{c} = \frac{b-x}{b} \Leftrightarrow x = \frac{bc}{b+c}$.

$S = 2Aria(\Delta ADF)$ și $Aria(\Delta ADF) = \frac{x \cdot x \cdot \sin A}{2} \overset{\sin A \le 2}{\le} \frac{x^2}{2} \Rightarrow S \le x^2 \le \frac{b^2c^2}{(b+c)^2}$.

Dar $\frac{bc}{(b+c)^2} \le \frac{1}{4} \Leftrightarrow (b-c)^2 \ge 0$.

În final $S \le \frac{bc}{4}$.

Problema302.

a)Arătați că $x^4 - x^3y - xy^3 + y^4 \ge 0$,oricare ar fi $x, y \in \mathbf{R}$.

b)Dacă triunghiul ABC este dreptunghic în A și AD este înălțime, atunci:

$$BC^2 - AB \cdot AC \ge AD^2.$$

Adriana Duță, Făgăraș,Concursul Laurențiu Duican, 1997

Soluție

a) $x^4 - x^3y - xy^3 + y^4 \ge 0 \Leftrightarrow (x-y)^2(x^2 + xy + y^2) \ge 0$, cu egalitate pentru $x = y$.

b)Avem $h = AD = \frac{AB \cdot AC}{BC} = \frac{bc}{a}$ și $a^2 = \sqrt{b^2 + c^2}$.

$$BC^2 - AB \cdot AC \ge AD^2 \Leftrightarrow a^2 - bc \ge \left(\frac{bc}{a}\right)^2 \Leftrightarrow b^4 - b^3c - bc^3 + c^4 \ge 0 \overset{a)}{\Leftrightarrow}$$

$(b-c)^2(b^2 + bc + c^2) \ge 0$, cu egalitate pentru $b = c$.

Problema303.

In acute ΔABC, H -ortocenter, AD, BE, CF -altidudes, then:

$$\sec A + \sec B + \sec C \le \frac{AH}{HD} + \frac{BH}{HE} + \frac{CH}{HF}.$$

Mathematical Inequalities 11/2024, George Apostolopoulos, Greece

Soluție.

Lema.

In acute ΔABC, H -ortocenter, AD, BE, CF -altitudes, then:

$$\sum \frac{AH}{HD} = \frac{p^2 - r^2 - 4Rr}{p^2 - (2R+r)^2} - 3.$$

Demonstrație.

Folosim $\frac{AH}{HD} = \tan B \tan C - 1$ și $\sum \tan B \tan C = \frac{p^2 - r^2 - 4Rr}{p^2 - (2R+r)^2}$.

Folosim **Lema** și $\sum \sec A = \sum \frac{1}{\cos A} = \frac{p^2 + r^2 - 4R^2}{p^2 - (2R+r)^2}$, inegalitatea se scrie:

$\frac{p^2 + r^2 - 4R^2}{p^2 - (2R+r)^2} \le \frac{p^2 - r^2 - 4Rr}{p^2 - (2R+r)^2} - 3 \Leftrightarrow 3p^2 \le 16R^2 + 8Rr + r^2$, vezi Gerretsen:
$p^2 \le 4R^2 + 4Rr + 3r^2$.

Remarca.

In acute ΔABC, H -ortocenter, AD, BE, CF -altitudes, then:

$$\left(\frac{AH}{HD}\right)^n + \left(\frac{BH}{HE}\right)^n + \left(\frac{CH}{HF}\right)^n \ge 3 \cdot 2^n, n \in \mathbf{N}.$$

Marin Chirciu

Soluție.

Pentru $n = 0$ se obține egalitatea3=3.

Pentru $n = 1$ se obține $\sum \frac{AH}{HD} \ge 6$, vezi Lema.

Pentru $n \ge 2$ se folosește inegalitatea lui Holder.

$$LHS = \sum \left(\frac{AH}{HD}\right)^n \overset{Holder}{\ge} \frac{\left(\sum \frac{AH}{HD}\right)^n}{3^{n-1}} \overset{Lema}{\ge} \frac{6^n}{3^{n-1}} = 3 \cdot 2^n = RHS.$$

Lema.

In acute ΔABC, H -ortocenter, AD, BE, CF -altitudes, then:

$$\sum \frac{AH}{HD} \geq 6 .$$

Demonstrație.

Folosim $\frac{AH}{HD} = \tan B \tan C - 1$ și $\sum \tan B \tan C = \frac{p^2 - r^2 - 4Rr}{p^2 - (2R+r)^2}$.

$$\sum \frac{AH}{HD} = \frac{p^2 - r^2 - 4Rr}{p^2 - (2R+r)^2} - 3 \overset{Gerretsen}{\geq} \frac{16Rr - 5r^2 - r^2 - 4Rr}{4R^2 + 4Rr + 3r^2 - (2R+r)^2} - 3 = \frac{12Rr - 6r^2}{2r^2} - 3 =$$

$$= \frac{3(R-r)}{r} \overset{Euler}{\geq} 6 .$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema304.

In ΔABC

$$\sum \frac{\cos A}{\cos \frac{B}{2} \cos \frac{C}{2}} \geq 2 .$$

RMM 11/2024,Nguyen Hung Cuong, Vietnam

Soluție

$$LHS = \sum \frac{\cos A}{\cos \frac{B}{2} \cos \frac{C}{2}} \overset{AM-GM}{\geq} \sum \frac{\cos A}{\frac{\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}}{2}} = \sum \frac{2\cos A}{\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}} =$$

$$= \sum \frac{4\cos A}{1 + \cos B + 1 + \cos C} = 4\sum \frac{\cos A}{2 + \cos B + \cos C} \overset{Lema}{\geq} 4 \cdot \frac{1}{2} = 2 = RHS .$$

Lema.

In ΔABC

$$\sum \frac{\cos A}{2 + \cos B + \cos C} \geq \frac{1}{2} .$$

Demonstrație

$$\sum \frac{\cos A}{2 + \cos B + \cos C} = \sum \frac{\cos^2 A}{2\cos A + \cos A \cos B + \cos A \cos C} \overset{CS}{\geq} \frac{\left(\sum \cos A\right)^2}{2\sum \cos A + 2\sum \cos B \cos C} =$$

$$=\frac{\left(\sum\cos A\right)^2}{2\sum\cos A+2\sum\cos B\cos C}\geq\frac{\left(1+\frac{r}{R}\right)^2}{2\left(1+\frac{r}{R}\right)+2\frac{Rr+r^2}{R^2}}=\frac{(R+r)^2}{2(R+r)^2}=\frac{1}{2}.$$

Remarca.

In ΔABC

$$\sum\frac{\cos A}{\lambda+\cos B+\cos C}\geq\frac{3}{2(\lambda+1)},0\leq\lambda\leq 2.$$

Marin Chirciu

Soluție

$$\sum\frac{\cos A}{\lambda+\cos B+\cos C}=\sum\frac{\cos^2 A}{\lambda\cos A+\cos A\cos B+\cos A\cos C}\overset{CS}{\geq}\frac{\left(\sum\cos A\right)^2}{\lambda\sum\cos A+2\sum\cos B\cos C}=$$

$$=\frac{\left(\sum\cos A\right)^2}{\lambda\sum\cos A+2\sum\cos B\cos C}\geq\frac{\left(1+\frac{r}{R}\right)^2}{\lambda\left(1+\frac{r}{R}\right)+2\frac{Rr+r^2}{R^2}}=\frac{R+r}{\lambda R+2r}\overset{Euler}{\geq}\frac{3}{2(\lambda+1)},$$

unde $\frac{R+r}{\lambda R+2r}\overset{Euler}{\geq}\frac{3}{2(\lambda+1)}\Leftrightarrow R(2-\lambda)\geq 2r(2-\lambda)$, vezi $R\geq 2r$ și $0\leq\lambda\leq 2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema306.

If $a,b,c>0$, $a^3+b^3+c^3=3$

$$(a+b)(b+c)(c+a)\leq 4+4abc.$$

THCS 11/2024, Nguyen Minh Tho, Vietnam

Soluție

Lema.

If $a,b,c>0$, $a^3+b^3+c^3=3$

$$(a+b)(b+c)(c+a)\leq 3+5abc.$$

Demonstrație

Folosim inegalitatea lui Schur $a^3+b^3+c^3+3abc \geq \sum ab(a+b)$.

$$(a+b)(b+c)(c+a)=\sum ab(a+b)+2abc \overset{Schur}{\leq} \sum a^3+3abc+2abc=3+5abc.$$

$$LHS=(a+b)(b+c)(c+a) \overset{Lema}{\leq} 3+5abc=3+4abc+abc \overset{abc\leq 1}{\leq} 3+4abc+1=4+4abc=RHS.$$

Am folosit mai sus $abc \leq 1$, care rezultă din $3=a^3+b^3+c^3 \geq 3abc \Rightarrow abc \leq 1$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $a^3+b^3+c^3=3$ then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{ab+bc+ca}{1+abc}.$$

Marin Chirciu

Soluție

$$LHS=\sum \frac{1}{a+b}=\frac{\sum(b+c)(c+a)}{\prod(a+b)}=\frac{\sum a^2+3\sum ab}{\prod(a+b)} \geq \frac{4\sum ab}{\prod(a+b)} \overset{Lema}{\geq} \frac{4\sum ab}{4(1+abc)}=RHS.$$

Lema.

If $a,b,c>0$, $a^3+b^3+c^3=3$

$$(a+b)(b+c)(c+a) \leq 4+4abc.$$

Demonstrație

Folosim inegalitatea lui Schur $a^3+b^3+c^3+3abc \geq \sum ab(a+b)$.

$$(a+b)(b+c)(c+a)=\sum ab(a+b)+2abc \overset{Schur}{\leq} \sum a^3+3abc+2abc=3+5abc=$$

$$=3+4abc+abc \overset{abc\leq 1}{\leq} 3+4abc+1=4+4abc.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema307.

In ΔABC

$$\sum m_a w_a \ge p^2.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Lema.

In ΔABC

$$m_a w_a \ge bc\cos^2\frac{A}{2}.$$

Demonstrație

Folosim inegalitatea lui Lascu $m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$,(Mircea Lascu, 1987).

$$\Rightarrow m_a w_a \overset{Lascu}{\ge} \frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cos\frac{A}{2} = bc\cos^2\frac{A}{2}.$$

$$LHS = \sum m_a w_a \overset{Lema}{\ge} \sum bc\cos^2\frac{A}{2} = p^2 = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ \sum m_a w_a \le p^2\left(\frac{R}{2r}\right)^2.$$

Soluție

Lema.

In ΔABC

$$m_a w_a \le \frac{R^2 p^2}{a^2}.$$

Demonstrație

$$\Rightarrow m_a w_a \overset{Panaitopol}{\le} \frac{Rp}{a}\cdot\frac{Rp}{a} = \frac{R^2 p^2}{a^2}.$$

$$LHS = \sum m_a w_a \overset{Lema}{\le} \sum \frac{R^2 p^2}{a^2} = R^2 p^2 \sum \frac{1}{a^2} \overset{Steinig}{\le} R^2 p^2 \cdot \frac{1}{4r^2} = p^2\left(\frac{R}{2r}\right)^2 = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2)\ p^2 \le \sum m_a w_a \le p^2 \left(\frac{R}{2r}\right)^2 .$$

$$3)\ \sum bc\cos^2\frac{A}{2} \ge 3\sum bc\sin^2\frac{A}{2} .$$

Dezvoltări, Marin Chirciu

Soluție

Folosind $\sum bc\cos^2\frac{A}{2} = p^2$ și $\sum bc\sin^2\frac{A}{2} = r(4R+r)$ inegalitatea se scrie:

$p^2 \ge 3\cdot r(4R+r)$, care rezultă din inegalitatea lui Gerretsen: $p^2 \ge 16Rr - 5r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema308.

In ΔABC

$$\sum \frac{\sin B\sin C}{\sin^2\frac{A}{2}} \ge 9 .$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Lema.

In ΔABC

$$\sum \frac{\sin B\sin C}{\sin^2\frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3(4R+r)}{4R^2r^2} .$$

$$LHS = \sum \frac{\sin B\sin C}{\sin^2\frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3(4R+r)}{4R^2r^2} \overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} \frac{\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-12Rr\right)+r^3(4R+r)}{4R^2r^2} = \frac{r^2\left(64R^2-64Rr+16r^2\right)}{4R^2r^2} =$$

$$= \frac{4\left(4R^2-4Rr+r^2\right)}{R^2} \overset{Euler}{\ge} 9 = RHS .$$

Remarca.

In ΔABC

$$1)\left(4-\frac{2r}{R}\right)^2 \le \sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}} \le 4\left(\frac{R^2}{r^2}-7\frac{r^2}{R^2}\right).$$

Soluție

Lema.

In ΔABC

$$\sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2}.$$

Inegalitatea din dreapta.

$$\sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2} \overset{Gerretsen}{\le}.$$

$$\overset{Gerretsen}{\le} \frac{\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2}=$$

$$=\frac{16R^4-16R^3r+16r^4}{4R^2r^2}=\frac{4\left(R^4-R^3r+r^4\right)}{R^2r^2} \overset{Euler}{\le} \frac{4\left(R^4-7r^4\right)}{R^2r^2}=4\left(\frac{R^2}{r^2}-7\frac{r^2}{R^2}\right).$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2} \overset{Gerretsen}{\ge}.$$

$$\overset{Gerretsen}{\ge} \frac{\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2}=\frac{r^2\left(64R^2-64Rr+16r^2\right)}{4R^2r^2}=$$

$$=\frac{4\left(4R^2-4Rr+r^2\right)}{R^2}=\frac{4\left(2R-r\right)^2}{R^2}=\left(4-\frac{2r}{R}\right)^2.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2)\ \frac{6r}{R} \le \sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} \le 3.$$

Soluție

Lema.

In ΔABC

$$\sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-4Rr\right)+r\left(4R+r\right)^3}{4R^2p^2}.$$

Inegalitatea din dreapta.

$$\sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-4Rr\right)+r\left(4R+r\right)^3}{4R^2p^2} = \frac{1}{4R^2}\left[p^2+2r^2-4Rr+\frac{r\left(4R+r\right)^3}{p^2}\right] \overset{Gerretsen}{\le}.$$

$$\overset{Gerretsen}{\le} \frac{1}{4R^2}\left[4R^2+4Rr+3r^2+2r^2-4Rr+\frac{r\left(4R+r\right)^3}{\frac{r\left(4R+r\right)^2}{R+r}}\right] = \frac{8R^2+5Rr+6r^2}{4R^2} \overset{Euler}{\le} 3.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-4Rr\right)+r\left(4R+r\right)^3}{4R^2p^2} = \frac{1}{4R^2}\left[p^2+2r^2-4Rr+\frac{r\left(4R+r\right)^3}{p^2}\right] \overset{Gerretsen}{\ge}.$$

$$\overset{Gerretsen}{\ge} \frac{1}{4R^2}\left[16Rr-5r^2+2r^2-4Rr+\frac{r\left(4R+r\right)^3}{\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}}\right] = \frac{r\left(28R^2-7Rr-2r^2\right)}{4R^3} \overset{Euler}{\ge} \frac{6r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3)\ 3\sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} \le \sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}}.$$

Dezvoltări, Marin Chirciu

Soluție

Lema1.

In ΔABC

$$\sum \frac{\sin B \sin C}{\cos^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-4Rr\right)+r\left(4R+r\right)^3}{4R^2p^2}.$$

Lema2.

In ΔABC

$$\sum \frac{\sin B \sin C}{\sin^2 \frac{A}{2}} = \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2}.$$

Folosind **Lemele** inegalitatea se scrie:

$$3\cdot \frac{p^2\left(p^2+2r^2-4Rr\right)+r\left(4R+r\right)^3}{4R^2p^2} \le \frac{p^2\left(p^2+2r^2-12Rr\right)+r^3\left(4R+r\right)}{4R^2r^2} \Leftrightarrow$$

$$\Leftrightarrow 3r^2p^2\left(p^2+2r^2-4Rr\right)+3r^3\left(4R+r\right)^3 \le p^4\left(p^2+2r^2-12Rr\right)+p^2r^3\left(4R+r\right) \Leftrightarrow$$

$$\Leftrightarrow p^6-p^4\left(r^2+12Rr\right)+p^2r^3\left(16R-5r\right) \ge 3r^3\left(4R+r\right)^3 \Leftrightarrow$$

$\Leftrightarrow p^2\left[p^2\left(p^2-r^2-12Rr\right)+r^3\left(16R-5r\right)\right] \ge 3r^3\left(4R+r\right)^3$, care rezultă din inegalitatea lui Gerretsen: $p^2 \ge 16Rr-5r^2 \ge \frac{r\left(4R+r\right)^2}{R+r}$.

Rămâne să arătăm că:

$$\frac{r\left(4R+r\right)^2}{R+r}\left[\left(16Rr-5r^2\right)\left(16Rr-5r^2-r^2-12Rr\right)+r^3\left(16R-5r\right)\right] \ge 3r^3\left(4R+r\right)^3 \Leftrightarrow$$

$$\Leftrightarrow \left(16R-5r\right)\left(4R-6r\right)+r\left(16R-5r\right) \ge 3\left(R+r\right)\left(4R+r\right) \Leftrightarrow 52R^2-115Rr+22r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow \left(R-2r\right)\left(52R-11r\right) \ge 0$, vezi $R \ge 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema309.

If $x, y > 0$

$$\frac{1}{x^2+y}+\frac{1}{y^2+x}\le\frac{x+y+2}{(x+y)^2}.$$

MathAtelier 11/2024, Panagiotis Danousis, Greece

Soluție

Lema.

If $x, y > 0$ then

$$\frac{1}{x^2+y}\le\frac{y+1}{(x+y)^2}.$$

Demonstrație

$\frac{1}{x^2+y}\le\frac{y+1}{(x+y)^2}\Leftrightarrow y(x-1)^2\ge 0$, cu egalitate pentru $x=1$.

$$LHS=\frac{1}{x^2+y}+\frac{1}{y^2+x}\overset{Lema}{\le}\frac{y+1}{(x+y)^2}+\frac{x+1}{(x+y)^2}=\frac{x+y+2}{(x+y)^2}=RHS$$

Remarca.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum\frac{1}{x^2+y}\le\frac{3}{2xyz}.$$

Marin Chirciu

Soluție

Lema.

If $x, y > 0$ then

$$\frac{1}{x^2+y}\le\frac{y+1}{4xy}.$$

$$LHS=\sum\frac{1}{x^2+y}\overset{Lema}{\le}\sum\frac{y+1}{4xy}=\frac{\sum yz+\sum x}{4xyz}\le\frac{3+3}{4xyz}=\frac{3}{2xyz}=RHS.$$

Am folosit $\sum yz \le 3$, vezi $\sum yz \overset{SOS}{\le} \frac{1}{3}\left(\sum x\right)^2 = \frac{1}{3}3^2 = 3$.

Problema310.

In ΔABC

$$\sum \frac{m_a}{\sin\frac{A}{2}} \ge \frac{a^2+b^2+c^2}{2r}.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Lema.

In ΔABC

$$\frac{m_a}{\sin\frac{A}{2}} \ge \frac{b+c}{2}\cot\frac{A}{2}.$$

Demonstrație

Folosim inegalitatea lui Lascu $m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$,(Mircea Lascu, 1987).

$$\Rightarrow \frac{m_a}{\sin\frac{A}{2}} \overset{Lascu}{\ge} \frac{\frac{b+c}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}} = \frac{b+c}{2}\cot\frac{A}{2}.$$

$$LHS = \sum \frac{m_a}{\sin\frac{A}{2}} \overset{Lema}{\ge} \sum \frac{b+c}{2}\cot\frac{A}{2} = \frac{1}{2}\sum (b+c)\cot\frac{A}{2} = \frac{a^2+b^2+c^2}{2r} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ \sum \frac{m_a \tan\frac{A}{2}}{\cos\frac{A}{2}} \ge 2(R+r).$$

Soluție

Lema.

In ΔABC

$$\frac{m_a \tan\frac{A}{2}}{\cos\frac{A}{2}} \geq \frac{b+c}{2}\tan\frac{A}{2}.$$

Demonstrație

Folosim inegalitatea lui Lascu $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$,(Mircea Lascu, 1987).

$$\Rightarrow \frac{m_a \tan\frac{A}{2}}{\cos\frac{A}{2}} \overset{Lascu}{\geq} \frac{\frac{b+c}{2}\cos\frac{A}{2}\tan\frac{A}{2}}{\cos\frac{A}{2}} = \frac{b+c}{2}\tan\frac{A}{2}.$$

$$LHS = \sum \frac{m_a \tan\frac{A}{2}}{\cos\frac{A}{2}} \overset{Lema}{\geq} \sum \frac{b+c}{2}\tan\frac{A}{2} = \frac{1}{2}\sum(b+c)\tan\frac{A}{2} = \frac{1}{2}\cdot 4(R+r) = 2(R+r) = RHS.$$

2) $3\sum(b+c)\tan\frac{A}{2} \leq \sum(b+c)\cot\frac{A}{2}$.

Dezvoltări, Marin Chirciu

Soluție

Folosind $\sum(b+c)\tan\frac{A}{2} = 4(R+r)$ și $\sum(b+c)\cot\frac{A}{2} = \frac{2(p^2-r^2-4Rr)}{r}$ obținem:

$3\cdot 4(R+r) \leq \frac{2(p^2-r^2-4Rr)}{r} \Leftrightarrow p^2 \geq 10Rr + 7r^2$, vezi Gerretsen: $p^2 \geq 16Rr - 5r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema311.

În ΔABC dreptunghic în A și $BC = 2AB$, fie BP bisectoarea unghiului B și $CD \perp BP, P \in (AC), D \in BP$.

Aflați valoarea raportului $\frac{AP}{PC}$.

Concursul Spiru Haret-Gheorghe Vrânceanu, 1994

Soluție

Fie O mijlocul lui BC, $BC = 2a$ Avem ΔABO echilateral de latură a.

$ABOD$ este romb și $ABCD$ trapez isoscel. Obținem $\frac{AP}{PC} = \frac{1}{2}$.

Problema312.

Aflați $\overline{xyz}$ asfel încât $\sqrt{4-x} + \sqrt{9-y} + \sqrt{24-z} + \sqrt{x+y+z} = 11$,

Concursul Spiru Haret-Gheorghe Vrânceanu, 1994

Soluție

Obținem $\overline{xyz} = 358$.

Problema313.

If $a > 1$ then

$$\log_a(a+1) > \log_{a+1}(a+2).$$

Manual X

Soluție

Folosim: $(a+1)^2 > a(a+2) \Leftrightarrow 1 > 0$.

$$\Rightarrow \frac{a+1}{a} > \frac{a+2}{a+1} \Rightarrow \log_a \frac{a+1}{a} > \log_a \frac{a+2}{a+1} \text{ și } \log_a x > \log_{a+1} x \Rightarrow$$

$$\Rightarrow \log_a \frac{a+1}{a} > \log_a \frac{a+2}{a+1} > \log_{a+1} \frac{a+2}{a+1} \Rightarrow \log_a \frac{a+1}{a} > \log_{a+1} \frac{a+2}{a+1} \Leftrightarrow$$

$$\Leftrightarrow \log_a(a+1) - \log_a a > \log_{a+1}(a+2) - \log_{a+1}(a+1) \Leftrightarrow \log_a(a+1) > \log_{a+1}(a+2).$$

Problema314.

In ΔABC

$$\sum \sqrt{\cot A} \sin A \le \frac{3}{4} \cdot \frac{R}{r} \sqrt[4]{3}.$$

RMM 11/2024, Konstantinos Geronikolas, Greece

Soluție

$$LHS=\sum\sqrt{\cot A}\sin A\overset{CBS}{\leq}\sqrt{\sum\cot A\sum\sin^2 A}\leq\sqrt{\frac{9R^2}{4pr}\cdot\frac{9}{4}}=\frac{9R}{4}\sqrt{\frac{1}{pr}}\overset{Mitrinovic}{\leq}\frac{3}{4}\cdot\frac{R}{r}\sqrt[4]{3}=RHS.$$

Am folosit mai sus: $\sum\cot A=\frac{p^2-r^2-4Rr}{2pr}\overset{Gerretsen}{\leq}\frac{9R^2}{4pr}$ și $\sum\sin^2 A=\frac{p^2-r^2-4Rr}{2R^2}\overset{Gerretsen}{\leq}\frac{9}{4}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In acute ΔABC

$$\frac{3\sqrt[6]{F\left(5r^2-R^2\right)}}{\sqrt[3]{2R^2}}\leq\sum\sqrt{\cot A}\sin A\leq\frac{9R}{4\sqrt{F}}.$$

Marin Chirciu

Soluție

Inegalitatea din dreapta.

$$\sum\sqrt{\cot A}\sin A\overset{CBS}{\leq}\sqrt{\sum\cot A\sum\sin^2 A}\leq\sqrt{\frac{9R^2}{4pr}\cdot\frac{9}{4}}=\frac{9R}{4}\sqrt{\frac{1}{pr}}=\frac{9R}{4\sqrt{F}}.$$

Am folosit mai sus: $\sum\cot A=\frac{p^2-r^2-4Rr}{2pr}\overset{Gerretsen}{\leq}\frac{9R^2}{4pr}$

Inegalitatea din stânga.

$$\sum\sqrt{\cot A}\sin A\overset{AM-GM}{\geq}3\sqrt[3]{\prod\sqrt{\cot A}\prod\sin A}\geq3\sqrt[3]{\sqrt{\frac{5r^2-R^2}{F}}\cdot\frac{F}{2R^2}}=3\cdot2^{\frac{-1}{3}}\left(5r^2-R^2\right)^{\frac{1}{6}}F^{\frac{1}{6}}R^{\frac{-2}{3}}=$$

$$=\frac{3\sqrt[6]{F\left(5r^2-R^2\right)}}{\sqrt[3]{2R^2}}.$$

Am folosit mai sus:

$$\prod\cot A=\frac{p^2-\left(2R+r\right)^2}{2pr}\overset{Walker}{\geq}\frac{2R^2+8Rr+3r^2-4R^2-4Rr-r^2}{2pr}=\frac{-2R^2+4Rr+2r^2}{2pr}=$$

$$=\frac{-R^2+2Rr+r^2}{pr}\overset{Euler}{\geq}\frac{5r^2-R^2}{pr}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema315.

In ΔABC cu $abc=1$

$$\sum\frac{b^2+c^2-a^2}{b+c-a}\le 3R\sum\cos A.$$

RMM 11/2024, Ertan Yildirim, Turkey

Soluție

Lema

In ΔABC

$$\sum\frac{b^2+c^2-a^2}{b+c-a}=\frac{5p^2-(4R+r)^2}{p}.$$

Folosind **Lema** și identitatea în triunghi $\sum\cot A=\frac{p^2-r^2-4Rr}{2pr}$, inegalitatea se scrie:

$$\frac{5p^2-(4R+r)^2}{p}\le 3R\cdot\frac{p^2-r^2-4Rr}{2pr}\Leftrightarrow p^2(3R-10r)+r\left(20R^2+13Rr+2r^2\right)\ge 0.$$

Cazul1). Dacă $(3R-10r)\ge 0$, inegalitatea este evidentă.

Cazul2). Dacă $(3R-10r)<0$, inegalitatea se rescrie:

$r\left(20R^2+13Rr+2r^2\right)\ge p^2(10r-3R)$, vezi Gerretsen: $p^2\le 4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\frac{2}{p}\left(59r^2-8R^2\right)\le\sum\frac{b^2+c^2-a^2}{b+c-a}\le\frac{27R^2}{2p}.$$

Marin Chirciu

Problema316.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{1}{x+1}\ge\frac{2\left(x^2+y^2+z^2\right)+21}{18}.$$

RMM 11/2024, Phan Ngoc Chau, Vietnam

Soluție

Folosim pqr -Method.

Notăm: $p = x + y + z = 3, q = xy + yz + zx, r = xyz$.

Avem $q \ge 3\sqrt[3]{r^2}$, vezi $q = xy + yz + zx \overset{AM-GM}{\ge} 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{r^2}$.

Inegalitatea se scrie:

$$\sum \frac{1}{x+1} \ge \frac{2(x^2+y^2+z^2)+21}{18} \Leftrightarrow \frac{\sum (y+1)(z+1)}{\prod (x+1)} \ge \frac{2(x^2+y^2+z^2)+21}{18} \Leftrightarrow$$

$$\Leftrightarrow \frac{q+2p+3}{r+q+p+1} \ge \frac{2(p^2-2q)+21}{18} \overset{p=3}{=} \frac{q+9}{r+q+4} \ge \frac{2(9-2q)+21}{18} \Leftrightarrow 4q^2 - 5q + 4qr + 6 \ge 39r,$$

care rezultă din $q \ge 3\sqrt[3]{r^2}$.

Rămâne să arătăm că:

$$3\sqrt[3]{r^2}\left(4\cdot 3\sqrt[3]{r^2} - 5\right) + 4\cdot 3\sqrt[3]{r^2}r + 6 \ge 39r \overset{\sqrt[3]{r}=t}{\Leftrightarrow} 3t^2\left(4\cdot 3t^2 - 5\right) + 4\cdot 3t^2\cdot t^3 + 6 \ge 39t^3 \Leftrightarrow$$

$\Leftrightarrow 12t^5 + 36t^4 - 39t^3 - 15t^2 + 6 \ge 0 \Leftrightarrow (t-1)(12t^4 + 48t^3 + 9t^2 - 6t - 6) \ge 0$???vezi

$$9 = (x+y+z)^2 \overset{SOS}{\ge} 3(xy+yz+zx) = 3q \Rightarrow 9 \ge 3q \Rightarrow q \le 3.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

In ΔABC

1) $$\sum \frac{1}{\frac{3r}{r_a}+2} \ge \frac{6}{\frac{3r(4R+r)}{p^2}+5}.$$

Soluție

Lema

If $x, y, z > 0$, $x + y + z = 3$ then

$$\sum \frac{1}{x+2} \ge \frac{18}{xy+yz+zx+15}.$$

Soluție

Folosim pqr -Method.

Notăm: $p = x + y + z = 3, q = xy + yz + zx, r = xyz$.

Avem $q^2 \ge 3rp$, vezi $q^2 = (xy + yz + zx)^2 \ge 3xyz(x + y + z) = 3rp$.

Inegalitatea se scrie:

$$\sum \frac{1}{x+2} \ge \frac{18}{xy+yz+zx+15} \Leftrightarrow \frac{\sum (y+2)(z+2)}{\prod (x+2)} \ge \frac{18}{xy+yz+zx+15} \Leftrightarrow$$

$$\Leftrightarrow \frac{q+4p+12}{r+2q+4p+8} \ge \frac{18}{q+15} \overset{p=3}{=} \frac{q+24}{r+2q+20} \ge \frac{18}{q+15} \Leftrightarrow q^2 + 3q \ge 18r$$, care rezultă din $r \le \frac{q^2}{9}$, vezi $q^2 \ge 3rp$, $p = 3 \Rightarrow q^2 \ge 9r$.

Rămâne să arătăm că:

$$q^2 + 3q \ge 18 \cdot \frac{q^2}{9} \Leftrightarrow q \le 3$$, vezi $9 = (x+y+z)^2 \overset{SOS}{\ge} 3(xy+yz+zx) = 3q \Rightarrow 9 \ge 3q \Rightarrow q \le 3$.

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Să trecem la rezolvarea problemei din enunț.

Se cunoaște identitatea în triunghi $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \Leftrightarrow \frac{3r}{r_a} + \frac{3r}{r_b} + \frac{3r}{r_c} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obținem:

$$\sum \frac{1}{\frac{3r}{r_a}+2} \ge \frac{18}{\frac{3r}{r_a}\cdot\frac{3r}{r_b}+\frac{3r}{r_b}\cdot\frac{3r}{r_c}+\frac{3r}{r_a}\cdot\frac{3r}{r_c}+15} \overset{(1)}{\Leftrightarrow} \sum \frac{1}{\frac{3r}{r_a}+2} \ge \frac{6}{\frac{3r^2}{r_a r_b}+\frac{3r^2}{r_b r_c}+\frac{3r^2}{r_c r_a}+5} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{3\frac{r}{r_a}+2} \ge \frac{6}{3r^2 \cdot \frac{4R+r}{rp^2}+5} \Leftrightarrow \sum \frac{1}{\frac{3r}{r_a}+2} \ge \frac{6}{\frac{3r(4R+r)}{p^2}+5}.$$

$$2)\ \sum \frac{1}{3\frac{r}{h_a}+2} \ge \frac{6}{\frac{3(R+r)^2}{p^2}+5}.$$

Dezvoltări, Marin Chirciu

Problema317.

If $x, y, z > 0$, $xyz = 1$ then

$$\sum \frac{x}{(x+2)(y+2)} \geq \frac{1}{3}.$$

RMM 11/2024, Tahirov Shirvan, Azerbaijan

Soluție

Folosim pqr -Method.

Notăm: $p = x + y + z, q = xy + yz + zx, r = xyz = 1$.

Avem $p \geq 3$, vezi $p = x + y + z \overset{AM-GM}{\geq} 3\sqrt[3]{xyz} = 3\sqrt[3]{1} = 3$.

Analog $q \geq 3$, vezi $q = xy + yz + zx \overset{AM-GM}{\geq} 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{1} = 3$.

Obținem:

$$LHS = \sum \frac{x}{(x+2)(y+2)} = \frac{\sum x(z+2)}{\prod(x+2)} = \frac{q+2p}{r+2q+4p+8} \overset{r=1}{=} \frac{q+2p}{2q+4p+9} \overset{(1)}{\geq} \frac{1}{3} = RHS,$$

unde $\frac{q+2p}{2q+4p+9} \overset{(1)}{\geq} \frac{1}{3} \Leftrightarrow 2p + q \geq 9$, care rezultă din $p \geq 3$, $q \geq 3$, vezi mai sus.

Remarca.

If $x, y, z > 0$, $xyz = 1$ and $\lambda \geq 0$ then

$$\sum \frac{x}{(x+\lambda)(y+\lambda)} \geq \frac{3}{(\lambda+1)^2}.$$

Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm: $p = x + y + z, q = xy + yz + zx, r = xyz = 1$.

Avem $p \geq 3$, vezi $p = x + y + z \overset{AM-GM}{\geq} 3\sqrt[3]{xyz} = 3\sqrt[3]{1} = 3$.

Analog $q \geq 3$, vezi $q = xy + yz + zx \overset{AM-GM}{\geq} 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{1} = 3$.

Obținem:

$$LHS=\sum\frac{x}{(x+\lambda)(y+\lambda)}=\frac{\sum x(z+\lambda)}{\prod(x+\lambda)}=\frac{q+\lambda p}{r+\lambda q+\lambda^2p+\lambda^3}\overset{r=1}{=}\frac{q+\lambda p}{\lambda q+\lambda^2p+\lambda^3+1}\overset{(1)}{\geq}$$
$$\overset{(1)}{\geq}\frac{3}{(\lambda+1)^2}=RHS\ ,$$

unde $\frac{q+\lambda p}{\lambda q+\lambda^2p+\lambda^3+1}\overset{(1)}{\geq}\frac{3}{(\lambda+1)^2}\Leftrightarrow \lambda p+q\geq 3(\lambda+1)$, care rezultă din $p\geq 3\,,q\geq 3$.

Problema318.

If $a,b,c>0\,,\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=3$ then

$$3abc\sum a^2b\sum ab^2\geq(ab+bc+ca)^3.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Avem $3=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}}=\frac{3}{abc}\Rightarrow abc\geq 1$.

Folosind $abc\geq 1$,rămâne să arătăm că: $3\sum a^2b\sum ab^2\geq(ab+bc+ca)^3$, care rezultă din:

$$\sum 1\sum a^2b\sum ab^2\overset{Holder}{\geq}\left(\sum\sqrt[3]{1\cdot a^2b\cdot ab^2}\right)^3=\left(\sum ab\right)^3.$$

Remarca.

If $a,b,c,d>0\,,\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}=4$ then

$$16abcd\sum a^3b\sum ab^3\geq(ab+bc+cd+da)^4.$$

Marin Chirciu

Soluție

Avem $4=\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}\overset{AM-GM}{\geq}4\sqrt[4]{\frac{1}{a^4}\cdot\frac{1}{b^4}\cdot\frac{1}{c^4}\cdot\frac{1}{d^4}}=\frac{4}{abcd}\Rightarrow abcd\geq 1$.

Folosind $abcd\geq 1$,rămâne să arătăm că: $16\sum a^2b\sum ab^2\geq(ab+bc+cd+da)^4$, vezi

$$\sum 1\sum 1\sum a^3b\sum ab^3 \overset{Holder}{\geq} \left(\sum \sqrt[4]{1\cdot 1\cdot a^3b\cdot ab^3}\right)^4 = \left(\sum ab\right)^4.$$

Problema319.

If $a,b,c>0$, $a+b+c=1$then:

$$\left(ab+bc+ca-abc\right)\sqrt[4]{\sum \frac{1}{a^3\left(b+c\right)^4}} \geq 1.$$

Mathematical Inequalities 11/2024,George Apostolopoulos, Greece

Soluție.

Folosim pqr -Method.

Notăm: $p=a+b+c=1, q=ab+bc+ca, r=abc$.

Avem $q^2 \geq 3rp$, vezi $q^2 = \left(ab+bc+ca\right)^2 \geq 3abc\left(a+b+c\right) = 3rp$, $p=1 \Rightarrow q^2 \geq 3r$.

$p^2 \geq 3q$, vezi $p^2 = \left(a+b+c\right)^2 \geq 3\left(ab+bc+ca\right) = 3q$, $p=1 \Rightarrow 1 \geq 3q$.

$$LHS = \left(ab+bc+ca-abc\right)\sqrt[4]{\sum \frac{1}{a^3\left(b+c\right)^4}} = \left(q-r\right)\sqrt[4]{\sum \frac{\frac{1}{\left(b+c\right)^4}}{a^3}} \overset{Radon}{\geq} \left(q-r\right)\sqrt[4]{\frac{\left(\sum \frac{1}{b+c}\right)^4}{\left(\sum a\right)^3}} =$$

$$\overset{\sum a=1}{=} \left(q-r\right)\sqrt[4]{\frac{\left(\sum \frac{1}{b+c}\right)^4}{1}} = \left(q-r\right)\sum \frac{1}{b+c} = \left(q-r\right)\frac{p^2+q}{pq-r} \overset{p=1}{=} \left(q-r\right)\frac{1+q}{q-r} \overset{(1)}{\geq} 1 = RHS,$$

unde $\left(q-r\right)\frac{1+q}{q-r} \overset{(1)}{\geq} 1 \Leftrightarrow q^2 \geq r\left(3q+2\right)$, care rezultă din $q^2 \geq 3r$.

Rămâne să arătăm că:

$3r \geq r\left(3q+2\right) \Leftrightarrow 3 \geq 3q+2 \Leftrightarrow 1 \geq 3q$, vezi mai sus.

Remarca.

If $a,b,c>0$, $a+b+c=1$and $n \in \mathbf{N}$ then:

$$\left(ab+bc+ca-abc\right)\sqrt[n+1]{\sum \frac{1}{a^n\left(b+c\right)^{n+1}}} \geq 1.$$

Marin Chirciu

Soluție.

Folosim pqr -Method.

Notăm: $p = a+b+c = 1, q = ab+bc+ca, r = abc$.

Avem $q^2 \geq 3rp$, vezi $q^2 = (ab+bc+ca)^2 \geq 3abc(a+b+c) = 3rp$, $p=1 \Rightarrow q^2 \geq 3r$.

$p^2 \geq 3q$, vezi $p^2 = (a+b+c)^2 \geq 3(ab+bc+ca) = 3q$, $p=1 \Rightarrow 1 \geq 3q$.

Obținem:

$$(ab+bc+ca-abc)\sqrt[n+1]{\sum \frac{1}{a^n(b+c)^{n+1}}} = (q-r)\sqrt[n+1]{\sum \frac{\frac{1}{(b+c)^{n+1}}}{a^n}} \overset{Radon}{\geq} (q-r)\sqrt[4]{\frac{\left(\sum \frac{1}{b+c}\right)^{n+1}}{\left(\sum a\right)^n}} \overset{\sum a=1}{=}$$

$$\overset{\sum a=1}{=} (q-r)\sqrt[n+1]{\frac{\left(\sum \frac{1}{b+c}\right)^{n+1}}{1}} = (q-r)\sum \frac{1}{b+c} = (q-r)\frac{p^2+q}{pq-r} \overset{p=1}{=} (q-r)\frac{1+q}{q-r} \overset{(1)}{\geq} 1 = RHS ,$$

unde $(q-r)\frac{1+q}{q-r} \overset{(1)}{\geq} 1 \Leftrightarrow q^2 \geq r(3q+2)$, care rezultă din $q^2 \geq 3r$.

Problema320.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \frac{1}{x+2} \geq \frac{18}{xy+yz+zx+15} .$$

RMM 11/2024, Phan Ngoc Chau, Vietnam

Soluție

Folosim pqr -Method.

Notăm: $p = x+y+z = 3, q = xy+yz+zx, r = xyz$.

Avem $q \geq 3\sqrt[3]{r^2}$, vezi $q = xy+yz+zx \overset{AM-GM}{\geq} 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{r^2}$.

Inegalitatea se scrie:

$$\sum \frac{1}{x+2} \geq \frac{18}{xy+yz+zx+15} \Leftrightarrow \frac{\sum (y+2)(z+2)}{\prod (x+2)} \geq \frac{18}{xy+yz+zx+15} \Leftrightarrow$$

$$\Leftrightarrow \frac{q+4p+12}{r+2q+4p+8} \geq \frac{18}{q+15} \overset{p=3}{=} \frac{q+24}{r+2q+20} \geq \frac{18}{q+15} \Leftrightarrow q^2+3q \geq 18r ,$$

care rezultă din $q \geq 3\sqrt[3]{r^2}$.

Rămâne să arătăm că:

$$\left(3\sqrt[3]{r^2}\right)^2 + 3\left(3\sqrt[3]{r^2}\right) \geq 18r \overset{\sqrt[3]{r}=t}{\Leftrightarrow} \left(3t^2\right)^2 + 3\left(3t^2\right) \geq 18t^3 \Leftrightarrow 9t^4 + 9t^2 \geq 18t^3 \Leftrightarrow 9t^2\left(t-1\right)^2 \geq 0 .$$

Remarca.

If $x, y, z > 0$, $x + y + z = 3$ and $\lambda \geq 0$, $3\left(2\lambda^3 + 6\lambda^2 + 2\lambda + 1\right) = k\left(4\lambda + 1\right)$ then

$$\sum \frac{1}{x+\lambda} \geq \frac{3(3+k)}{(\lambda+1)(xy+yz+zx+k)} .$$

Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm: $p = x + y + z = 3, q = xy + yz + zx, r = xyz$.

Avem $q \geq 3\sqrt[3]{r^2}$, vezi $q = xy + yz + zx \overset{AM-GM}{\geq} 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{r^2}$.

Inegalitatea se scrie:

$$\sum \frac{1}{x+\lambda} \geq \frac{3(3+k)}{(\lambda+1)(xy+yz+zx+k)} \Leftrightarrow \frac{\sum (y+\lambda)(z+\lambda)}{\prod (x+\lambda)} \geq \frac{3(3+k)}{(\lambda+1)(xy+yz+zx+k)} \Leftrightarrow$$

$$\Leftrightarrow \frac{q+2\lambda p+3\lambda^2}{r+\lambda q+\lambda^2 p+\lambda^3} \geq \frac{3(3+k)}{(\lambda+1)(q+k)} \overset{p=3}{=} \frac{q+6\lambda+3\lambda^2}{r+\lambda q+3\lambda^2+\lambda^3} \geq \frac{3(3+k)}{(\lambda+1)(q+k)} \Leftrightarrow$$

$$(\lambda+1)q^2 + \left(3\lambda^3 + 9\lambda^2 - 3\lambda - 2k\lambda + k\right)q \geq 3(k+3)r + 9\lambda^3 + 27\lambda^2 - 6k\lambda ,$$

care rezultă din $q \geq 3\sqrt[3]{r^2}$.

Rămâne să arătăm că:

$$(\lambda+1)\left(3\sqrt[3]{r^2}\right)^2 + \left(3\lambda^3 + 9\lambda^2 - 3\lambda - 2k\lambda + k\right)\left(3\sqrt[3]{r^2}\right) \geq 3(k+3)r + 9\lambda^3 + 27\lambda^2 - 6k\lambda$$

$$\overset{\sqrt[3]{r}=t}{\Leftrightarrow} (\lambda+1)\left(3t^2\right)^2 + \left(3\lambda^3 + 9\lambda^2 - 3\lambda - 2k\lambda + k\right)\left(3t^2\right) \geq 3(k+3)t^3 + 9\lambda^3 + 27\lambda^2 - 6k\lambda \Leftrightarrow$$

$$9(\lambda+1)t^4 - 3(k+3)t^3 + 3\left(3\lambda^3 + 9\lambda^2 - 3\lambda - 2k\lambda + k\right)t^2 - 9\lambda^3 - 27\lambda^2 + 6k\lambda \geq 0 \Leftrightarrow$$

$(t-1)^2\left[9(\lambda+1)t^2+3(6\lambda-k+3)t+3\lambda^3+9\lambda^2+6\lambda-2\lambda k-k+3\right]\geq 0.\Leftrightarrow q\leq 3$, vezi

$9=(x+y+z)^2\overset{SOS}{\geq}3(xy+yz+zx)=3q\Rightarrow 9\geq 3q\Rightarrow q\leq 3$.

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Remarca.

In ΔABC

1) $$\sum\frac{1}{\frac{3r}{r_a}+2}\geq\frac{6}{\frac{3r(4R+r)}{p^2}+5}.$$

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{1}{x+2}\geq\frac{18}{xy+yz+zx+15}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\frac{1}{\frac{3r}{r_a}+2}\geq\frac{18}{\frac{3r}{r_a}\cdot\frac{3r}{r_b}+\frac{3r}{r_b}\cdot\frac{3r}{r_c}+\frac{3r}{r_a}\cdot\frac{3r}{r_c}+15}\Leftrightarrow\sum\frac{1}{\frac{3r}{r_a}+2}\geq\frac{6}{\frac{3r(4R+r)}{p^2}+5}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

2) $$\sum\frac{1}{3\frac{r}{h_a}+2}\geq\frac{6}{\frac{3(R+r)^2}{p^2}+5}.$$

Dezvoltări, Marin Chirciu

Problema321.

Aflați $\overline{abc}$ asfel încât $\overline{abc}=(a+1)(b+2)(c+3)$

MathOlymp 11/2024, Elton Papanikolla

Soluție

Obținem $\overline{abc} = 396$

Problema322.

Aflați $\overline{abc}$ asfel încât $\overline{abc}$ împărțit la $\overline{cba}$ să dea câtul5 ți restul46

IneMath11/2024

Soluție

Folosim teorema împărțirii cu rest:

$\overline{abc} = \overline{cba} \cdot 5 + 46 \Leftrightarrow 100a + 10b + c = (100c + 10b + a) \cdot 5 + 46 \Leftrightarrow 95a = 539c + 126$.

În mod necesar $a \geq 7$.

Obținem $\overline{abc} = 731$

Problema323.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum x\sqrt{2y+1} \leq \sqrt{2 - x^2 - y^2 - z^2}\ .$$

MathOlymp 11/2024, Elton Papanikolla

Soluție

$$LHS = \sum x\sqrt{2y+1} = \sum \sqrt{x}\sqrt{2xy+x} \overset{CBS}{\leq} \sqrt{\sum x \sum (2xy+x)} \overset{\sum x=1}{=} \sum \sqrt{1 + 2\sum xy} =$$

$$= \sum \sqrt{1 + \left(\sum x\right)^2 - \sum x^2} = \sqrt{2 - \sum x^2} = RHS\ .$$

Remarca.

If $x, y, z > 0$, $x + y + z = 1$ and $\lambda \geq 0$ then

$$\sum x\sqrt{\lambda y+1} \leq \sum \sqrt{1 + \frac{\lambda}{2}\left(1 - \sum x^2\right)}\ .$$

Marin Chirciu

Soluție

$$LHS = \sum x\sqrt{\lambda y+1} = \sum \sqrt{x}\sqrt{\lambda xy+x} \overset{CBS}{\leq} \sqrt{\sum x \sum (\lambda xy+x)} \overset{\sum x=1}{=} \sum \sqrt{1 + \lambda\sum xy} =$$

$$=\sum\sqrt{1+\lambda\cdot\frac{1}{2}\left(\left(\sum x\right)^2-\sum x^2\right)}=\sum\sqrt{1+\frac{\lambda}{2}\left(1-\sum x^2\right)}=RHS.$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Remarca.

In ΔABC

1) $$\sum\frac{r}{r_a}\sqrt{2\frac{r}{r_b}+1}\leq\sqrt{2-\frac{1}{3}\left(\frac{2r}{R}\right)^2}.$$

Soluție

Lema

If $x,y,z>0$, $x+y+z=1$ then

$$\sum x\sqrt{2y+1}\leq\sqrt{2-x^2-y^2-z^2}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{r}{r_a}\sqrt{2\frac{r}{r_b}+1}\overset{Lema}{\leq}\sqrt{2-\sum\frac{r^2}{r_a^2}}\overset{(1)}{\Leftrightarrow}\sum\frac{r}{r_a}\sqrt{2\frac{r}{r_b}+1}\leq\sqrt{2-\frac{4r^2}{3R^2}}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{r}{r_a}\sqrt{2\frac{r}{r_b}+1}\leq\sqrt{2-\frac{1}{3}\left(\frac{2r}{R}\right)^2}.$$

2) $$\sum\frac{r}{h_a}\sqrt{2\frac{r}{h_b}+1}\leq\sqrt{\frac{5}{3}}.$$

Dezvoltări, Marin Chirciu

Problema324.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{1}{x^2+y+z}\leq 1.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\frac{1}{x^2+y+z}\le\frac{1+y+z}{9}.$$

Demonstrație

$$\left(x^2+y+z\right)\left(1+y+z\right)\overset{CBS}{\ge}\left(x+y+z\right)^2=3^2=9.$$

$$LHS=\sum\frac{1}{x^2+y+z}\overset{Lema}{\le}\sum\frac{1+y+z}{9}=\frac{3+2\sum x}{9}=\frac{3+2\cdot 3}{9}=1=RHS.$$

Remarca.

In ΔABC

$$1)\quad \sum\frac{1}{\frac{3r}{r_a^2}+\frac{1}{r_b}+\frac{1}{r_c}}\le 3r.$$

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{1}{x^2+y+z}\le 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $\left(x,y,z\right)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\frac{1}{\frac{9r^2}{r_a^2}+\frac{3r}{r_b}+\frac{3r}{r_c}}\le 1\Leftrightarrow\sum\frac{1}{\frac{3r}{r_a^2}+\frac{1}{r_b}+\frac{1}{r_c}}\le 3r.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2)\quad \sum\frac{1}{\frac{3r}{h_a^2}+\frac{1}{h_b}+\frac{1}{h_c}}\le 3r.$$

Dezvoltări, Marin Chirciu

Problema325.

If $x,y,z>0$, $xyz=1$ then

$$\frac{x+y+z}{3}\geq\sqrt[5]{\frac{x^2+y^2+z^2}{3}}.$$

Mathematical Inequalities 11/2024

Soluție

$$\frac{x+y+z}{3}\geq\sqrt[5]{\frac{x^2+y^2+z^2}{3}}\Leftrightarrow\left(\frac{x+y+z}{3}\right)^5\geq\frac{x^2+y^2+z^2}{3}\Leftrightarrow(x+y+z)^5\geq 81\left(x^2+y^2+z^2\right)$$

$\overset{xyz=1}{\Leftrightarrow}(x+y+z)^5\geq 81xyz\left(x^2+y^2+z^2\right)$, care rezultă din **Lema**.

Lema.

If $x,y,z>0$ then

$$(x+y+z)^5\geq 81xyz\left(x^2+y^2+z^2\right).$$

Demonstratie.

Fie $p=x+y+z, q=xy+yz+zx, r=xyz$.

Avem $xyz\left(x^2+y^2+z^2\right)=r\left(p^2-2q\right)\overset{3pr\leq q^2}{\leq}\frac{q^2\left(p^2-2q\right)}{3p}\overset{AGM}{\leq}\frac{1}{3p}\left(\frac{q+q+p^2-2q}{3}\right)^3=\frac{1}{3p}\cdot\frac{p^6}{27}=$

$$=\frac{p^5}{81}=\frac{(x+y+z)^5}{81}.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Problema326.

In ΔABC

$$\prod\left(1+\cot^2\frac{A}{2}\right)\geq 64.$$

RMM 11/2024, Neculai Stanciu, Buzău

Soluție

Lema.

In ΔABC

$$\prod\left(1+\cot^2\frac{A}{2}\right)=\left(\frac{4R}{r}\right)^2.$$

$$LHS=\prod\left(1+\cot^2\frac{A}{2}\right)=\left(\frac{4R}{r}\right)^2\overset{Euler}{\geq}64=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\quad \frac{64}{27}\leq\prod\left(1+\tan^2\frac{A}{2}\right)\leq\frac{16R^2}{27r^2}.$$

Soluție

Lema.

In ΔABC

$$\prod\left(1+\tan^2\frac{A}{2}\right)=\left(\frac{4R}{p}\right)^2.$$

Inegalitatea din dreapta.

$$\prod\left(1+\tan^2\frac{A}{2}\right)=\left(\frac{4R}{p}\right)^2=\frac{16R^2}{p^2}\overset{Mitrinovic}{\leq}\frac{16R^2}{27r^2}.$$

Inegalitatea din stânga.

$$\prod\left(1+\tan^2\frac{A}{2}\right)=\left(\frac{4R}{p}\right)^2=\frac{16R^2}{p^2}\overset{Mitrinovic}{\geq}\frac{16R^2}{\frac{27R^2}{4}}=\frac{64}{27}$$

ΔABC

$$2)\quad 27\prod\left(1+\tan^2\frac{A}{2}\right)\leq\prod\left(1+\cot^2\frac{A}{2}\right).$$

Dezvoltări, Marin Chirciu

Soluție

Lema1.

In ΔABC

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Lema2.

In ΔABC

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Folosind $\prod\left(1+\tan^2\frac{A}{2}\right)=\left(\frac{4R}{p}\right)^2$ și $\prod\left(1+\cot^2\frac{A}{2}\right)=\left(\frac{4R}{r}\right)^2$ negalitatea se scrie:

$27\left(\frac{4R}{p}\right)^2\le\left(\frac{4R}{r}\right)^2 \Leftrightarrow p^2\ge 27r^2$,(Mitrinovic).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema327.

If $x,y,z>0$, $xy+yx+zx=3$ then

$$\left(x+y+z\right)^4\ge 24\left(x^2+y^2+z^2\right).$$

Mathematical Inequalities 11/2024, Nguyen Minh Tho, Vietnam

Soluție

Folosim pqr -Method.

Notăm $p=x+y+z=3, q=xy+yz+zx=3, r=xyz$.

Avem: $x^2+y^2+z^2=p^2-2q, q=3 \Rightarrow x^2+y^2+z^2=p^2-6$.

Inegalitatea se scrie:

$p^4\ge 24\left(p^2-6\right)\Leftrightarrow p^4-24p^2+144\ge 0\Leftrightarrow\left(p^2-12\right)^2\ge 0$, cu egalitate pentru $p^2=12\Leftrightarrow$

$\Leftrightarrow p=2\sqrt{3}$.

Egalitatea are loc dacă și numai dacă $xy+yx+zx=3$ și $x+y+z=2\sqrt{3}$.

Problema328.

If ΔABC, G -centroid, K,L,M -mijloacele laturilor BC,CA,AB și A',B',C' -punctele unde medianele AK,BL,CP intersectează cercul circumscris ΔABC . Arătați că:

$$\frac{KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}}{\sin 2A+\sin 2B+\sin 2C}\ge R.$$

Mathematical Inequalities 11/2024, George Apostolopoulos, Greece

Soluție.

Lema

If ΔABC, G -centroid, K, L, M -mijloacele laturilor BC, CA, AB și A', B', C' -punctele unde medianele AK, BL, CP intersectează cercul circumscris ΔABC. Arătați că:

$$KA' \cdot \cot\frac{A}{2} + LB' \cdot \cot\frac{B}{2} + MC' \cdot \cot\frac{C}{2} \geq \frac{p^2(2R+3r) - r(4R+r)^2}{2Rp}.$$

Demonstrație.

Se scrie puterea punctului K față de cercul circumscris $\Delta ABC \Rightarrow$

$$\Rightarrow KA \cdot KA' = KB \cdot KC \Rightarrow m_a \cdot KA' = \frac{a}{2} \cdot \frac{a}{2} \Rightarrow KA' = \frac{a^2}{4m_a} \Rightarrow KA' \cdot \cot\frac{A}{2} = \frac{a^2}{4m_a} \cdot \cot\frac{A}{2}.$$

$$\sum KA' \cdot \cot\frac{A}{2} = \sum \frac{a^2}{4m_a} \cdot \cot\frac{A}{2} \overset{Panaitopol}{\geq} \sum \frac{a^2}{4 \cdot \frac{Rp}{a}} \cdot \cot\frac{A}{2} = \frac{1}{4Rp} \sum a^3 \cot\frac{A}{2} =$$

$$\frac{1}{4Rp} \cdot 2\left[p^2(2R+3r) - r(4R+r)^2\right] = \frac{p^2(2R+3r) - r(4R+r)^2}{2Rp}.$$

Am folosit mai sus: $\sum a^3 \cot\frac{A}{2} = 2\left[p^2(2R+3r) - r(4R+r)^2\right]$.

Inegalitatea lui Panaitopol $m_a \leq \frac{Rp}{a}$.

Folosind **Lema** și identitatea în triunghi $\sum \sin 2A = \frac{2pr}{R^2}$ obținem:

$$LHS = \frac{\sum KA' \cdot \cot\frac{A}{2}}{\sum \sin 2A} \overset{Lema}{\geq} \frac{\frac{p^2(2R+3r) - r(4R+r)^2}{2Rp}}{\frac{2pr}{R^2}} = \frac{R}{4rp^2}\left[p^2(2R+3r) - r(4R+r)^2\right] =$$

$$= \frac{R}{4r}\left[2R+3r - \frac{r(4R+r)^2}{p^2}\right] \overset{Gerretsen}{\geq} \frac{R}{4r}\left[2R+3r - \frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{R}{4r}(2R+3r-R-r) =$$

$$=\frac{R}{4r}\left(R+2r\right)\overset{Euler}{\geq}R=RHS.$$

Egalitatea are loc dacă și numai dacă triunghil este echilateral.

Remarca.

If ΔABC, G -centroid, K,L,M -mijloacele laturilor BC,CA,AB și A',B',C' -punctele unde medianele AK,BL,CP intersectează cercul circumscris ΔABC. Arătați că:

1) $$\frac{KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}}{\sin 2A+\sin 2B+\sin 2C}\leq 2r\left(\frac{R}{2r}\right)^5.$$

Soluție.

Lema

$$KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}\leq\frac{27R^3}{16rp}.$$

2) $$R\leq\frac{KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}}{\sin 2A+\sin 2B+\sin 2C}\leq 2r\left(\frac{R}{2r}\right)^5.$$

Remarca.

3) $$\frac{R}{2}\left(\frac{R}{2r}+1\right)\leq\frac{KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}}{\sin 2A+\sin 2B+\sin 2C}\leq 2r\left(\frac{R}{2r}\right)^5.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

$$KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}\leq\frac{27R^3}{16rp}.$$

$$LHS=\frac{\sum KA'\cdot\cot\frac{A}{2}}{\sum\sin 2A}\overset{Lema}{\leq}\frac{\frac{27R^3}{8rp}}{\frac{2pr}{R^2}}=\frac{27R^5}{16r^2p^2}\overset{Mitrinovic}{\leq}\frac{27R^5}{16r^2\cdot 27r^2}=\frac{R^5}{16r^4}=2r\left(\frac{R}{2r}\right)^5.$$

Inegalitatea din stânga.

Lema

$$KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}\ge\frac{p^2(2R+3r)-r(4R+r)^2}{2Rp}.$$

$$LHS=\frac{\sum KA'\cdot\cot\frac{A}{2}}{\sum\sin 2A}\overset{Lema}{\ge}\frac{\frac{p^2(2R+3r)-r(4R+r)^2}{2Rp}}{\frac{2pr}{R^2}}=\frac{R}{4rp^2}\left[p^2(2R+3r)-r(4R+r)^2\right]=$$

$$=\frac{R}{4r}\left[2R+3r-\frac{r(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\ge}\frac{R}{4r}\left[2R+3r-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{R}{4r}(2R+3r-R-r)=$$

$$=\frac{R}{4r}(R+2r)=\frac{R}{2}\left(\frac{R}{2r}+1\right)=RHS.$$

Egalitatea are loc dacă și numai dacă triunghil este echilateral.

4) $$\frac{27r^2}{p}\le KA'\cdot\cot\frac{A}{2}+LB'\cdot\cot\frac{B}{2}+MC'\cdot\cot\frac{C}{2}\le\frac{27R^3}{8rp}.$$

5) $$\frac{9Rr}{2p}\le KA'\cdot\tan\frac{A}{2}+LB'\cdot\tan\frac{B}{2}+MC'\cdot\tan\frac{C}{2}\le\frac{2R^3-7r^3}{rp}.$$

Dezvoltări, Marin Chirciu

Problema329.

Solve in real numbers

$$(1+x)\sqrt{1-x}+(1-x)\sqrt{1+x}=2\sqrt{1+x^2}.$$

Math Atelier 11/2024, Konstantinos Geronikolas, Greece

Soluție

Domeniul de definiție este $[-1,1]$.

Folosind CBS obținem:

$$2\sqrt{1+x^2}=(1+x)\sqrt{1-x}+(1-x)\sqrt{1+x}=\sqrt{1+x}\sqrt{1-x^2}+\sqrt{1-x}\sqrt{1-x^2}=$$

$$=\sqrt{1-x^2}\left(\sqrt{1+x}+\sqrt{1-x}\right)\overset{CBS}{\le}\sqrt{1-x^2}\sqrt{2(1+x+1-x)}=\sqrt{1-x^2}\cdot 2=2\sqrt{1-x^2}.$$

Rezultă $2\sqrt{1+x^2}\le 2\sqrt{1-x^2}\Leftrightarrow 1+x^2\le 1-x^2\Leftrightarrow 2x^2\le 0\Leftrightarrow x=0$.

Deducem că ecuația admite soluția unică $x = 0$.

Remarca.

Let $\lambda > 0$ fixed. Solve in real numbers

$$(\lambda + x)\sqrt{\lambda - x} + (\lambda - x)\sqrt{\lambda + x} = 2\sqrt{\lambda^3 + \lambda x^2}.$$

Marin Chirciu

Soluție

Domeniul de definiție este $[-\lambda, \lambda]$.

Folosind CBS obținem:

$$2\sqrt{\lambda^3 + \lambda x^2} = (\lambda + x)\sqrt{\lambda - x} + (\lambda - x)\sqrt{\lambda + x} = \sqrt{\lambda + x}\sqrt{\lambda^2 - x^2} + \sqrt{\lambda - x}\sqrt{\lambda^2 - x^2} =$$

$$= \sqrt{\lambda^2 - x^2}\left(\sqrt{\lambda + x} + \sqrt{\lambda - x}\right) \overset{CBS}{\leq} \sqrt{\lambda^2 - x^2}\sqrt{2(\lambda + x + \lambda - x)} = \sqrt{\lambda^2 - x^2} \cdot 2\sqrt{\lambda} =$$

$$= 2\sqrt{\lambda^3 - \lambda x^2}.$$

Rezultă $2\sqrt{\lambda^3 + \lambda x^2} \leq 2\sqrt{\lambda^3 - \lambda x^2} \Leftrightarrow \lambda^3 + \lambda x^2 \leq \lambda^3 - \lambda x^2 \Leftrightarrow 2\lambda x^2 \leq 0 \Leftrightarrow x = 0$.

Deducem că ecuația admite soluția unică $x = 0$.

Remarca.

Solve in real numbers

$$(2 + x)\sqrt{2 - x} + (2 - x)\sqrt{2 + x} = 2\sqrt{8 + 2x^2}.$$

Soluție

Cazul $\lambda = 2$.

Ecuația admite soluția unică $x = 0$.

Problema330.

If $x, y, z > 0$, $x + y + z = 3$ then

$$xyz\left(x^2 + y^2 + z^2\right) \leq 3.$$

Mathematical Inequalities 11/2024

Soluție

Folosim pqr -Method.

Notăm $p = x + y + z = 3, q = xy + yz + zx, r = xyz$.

Avem $q^2 \geq 3rp$,(1), vezi $q^2 = (xy + yz + zx)^2 \geq 3xyz(x + y + z) = 3rp$.

$$9 = p^2 = (x + y + z)^2 = \sum x^2 + 2\sum xy = \sum x^2 + 2q = \sum x^2 + q + q \overset{AM-GM}{\geq} 3\sqrt[3]{\sum x^2 \cdot q \cdot q} =$$

$$= 3\sqrt[3]{\sum x^2 \cdot q^2} \Rightarrow 9 \geq 3\sqrt[3]{\sum x^2 \cdot q^2} \Rightarrow \sum x^2 \leq \frac{27}{q^2}, (2).$$

Obținem:

$$LHS = xyz\left(x^2 + y^2 + z^2\right) = r\sum x^2 \overset{(2)}{\leq} r \cdot \frac{27}{q^2} \overset{p=3}{=} 3 \cdot \frac{3pr}{q^2} \overset{(1)}{\leq} 3 = RHS .$$

Remarca.

If $x, y, z > 0$, $x + y + z = 3$ and $n \in \mathbf{N}$ then

$$xyz\left(x^2 + y^2 + z^2\right)^n \leq 3^n .$$

Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm $p = x + y + z = 3, q = xy + yz + zx, r = xyz$.

$$3 = x + y + z \geq 3\sqrt[3]{xyz} = 3\sqrt[3]{r} \Rightarrow r \leq 1.$$

Avem $q^2 \geq 3rp$,(1), vezi $q^2 = (xy + yz + zx)^2 \geq 3xyz(x + y + z) = 3rp$.

$$9 = p^2 = (x + y + z)^2 = \sum x^2 + 2\sum xy = \sum x^2 + 2q = \sum x^2 + q + q \overset{AM-GM}{\geq} 3\sqrt[3]{\sum x^2 \cdot q \cdot q} =$$

$$= 3\sqrt[3]{\sum x^2 \cdot q^2} \Rightarrow 9 \geq 3\sqrt[3]{\sum x^2 \cdot q^2} \Rightarrow \sum x^2 \leq \frac{27}{q^2}, (2).$$

$$LHS = xyz\left(x^2 + y^2 + z^2\right)^n = r\left(\sum x^2\right)^n \overset{(2)}{\leq} r \cdot \left(\frac{27}{q^2}\right)^n \overset{p=3}{=} r \cdot \left(3 \cdot \frac{3pr}{q^2}\right)^n \overset{(1)}{\leq} r \cdot 3^n \overset{r \leq 1}{\leq} 3^n = RHS .$$

Remarca.

In ΔABC

$$1). \frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\le\frac{p^2}{81r^4}.$$

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$xyz\left(x^2+y^2+z^2\right)\le 3.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\frac{3r}{r_a}\cdot\frac{3r}{r_b}\cdot\frac{3r}{r_c}\left(\frac{9r^2}{r_a^2}+\frac{9r^2}{r_b^2}+\frac{9r^2}{r_c^2}\right)\le 3\Leftrightarrow\Leftrightarrow\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\le\frac{p^2}{81r^4}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2). \frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\le\frac{p^2}{81r^4}.$$

Dezvoltări, Marin Chirciu

Problema331.

If $a,b,c>0$, $ab+bc+ca=1$ then

$$\sum\frac{a^8}{\left(a^2+b^2\right)^2}\ge\frac{1}{12}.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

$$LHS=\sum\frac{a^8}{\left(a^2+b^2\right)^2}\overset{CS}{\ge}\sum\frac{a^8}{2\left(a^4+b^4\right)}\overset{CS}{\ge}\frac{1}{2}\frac{\left(\sum a^4\right)^2}{\sum\left(a^4+b^4\right)}=\frac{1}{2}\frac{\left(\sum a^4\right)^2}{2\sum a^4}=\frac{1}{4}\sum a^4=$$

$$\overset{CS}{\ge}\frac{1}{4}\cdot\frac{1}{3}\left(\sum a^2\right)^2\overset{CS}{\ge}\frac{1}{12}\left(\sum ab\right)^2=\frac{1}{12}(1)^2=\frac{1}{12}=RHS.$$

Remarca.

If $a,b,c>0$, $ab+bc+ca=1$ and $\lambda\ge 0$ then

$$1)\quad \sum\frac{a^8}{\left(a^2+\lambda b^2\right)^2}\ge\frac{1}{3(\lambda+1)^2}.$$

Soluție

$$LHS=\sum\frac{a^8}{\left(a^2+\lambda b^2\right)^2}\overset{CS}{\ge}\sum\frac{a^8}{(\lambda+1)\left(a^4+\lambda b^4\right)}\overset{CS}{\ge}\frac{1}{\lambda+1}\frac{\left(\sum a^4\right)^2}{\sum\left(a^4+\lambda b^4\right)}=\frac{1}{\lambda+1}\frac{\left(\sum a^4\right)^2}{(\lambda+1)\sum a^4}=$$

$$\frac{1}{(\lambda+1)^2}\sum a^4=\ge\frac{1}{(\lambda+1)^2}\cdot\frac{1}{3}\left(\sum a^2\right)^2\overset{CS}{\ge}\frac{1}{3(\lambda+1)^2}\left(\sum ab\right)^2=\frac{1}{3(\lambda+1)^2}(1)^2=\frac{1}{3(\lambda+1)^2}=$$

$$=RHS.$$

Am folosit mai sus $(\lambda+1)\left(a^4+\lambda b^4\right)\ge\left(a^2+\lambda b^2\right)^2\Leftrightarrow\lambda\left(a^2-b^2\right)^2\ge 0$.

Remarca.

If $a,b,c>0$, $ab+bc+ca=1$ and $\lambda\ge 0$ then

$$\sum\frac{a^4}{(a+\lambda b)^2}\ge\frac{1}{(\lambda+1)^2}.$$

Dezvoltări, Marin Chirciu

Soluție

$$LHS=\sum\frac{a^4}{(a+\lambda b)^2}\overset{CS}{\ge}\sum\frac{a^4}{(\lambda+1)\left(a^2+\lambda b^2\right)}\overset{CS}{\ge}\frac{1}{\lambda+1}\frac{\left(\sum a^2\right)^2}{\sum\left(a^2+\lambda b^2\right)}=\frac{1}{\lambda+1}\frac{\left(\sum a^2\right)^2}{(\lambda+1)\sum a^2}=$$

$$=\frac{1}{(\lambda+1)^2}\sum a^2\overset{CS}{\ge}\frac{1}{(\lambda+1)^2}\left(\sum ab\right)=\frac{1}{(\lambda+1)^2}\cdot 1=\frac{1}{(\lambda+1)^2}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=\frac{1}{\sqrt{3}}$.

Problema332.

If $a,b>0$ then

$$\frac{2ab}{a^2+b^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2\left(a^2+b^2\right)}\ge 5.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

$$LHS=\frac{2ab}{a^2+b^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2\left(a^2+b^2\right)}\overset{AM-GM}{\geq}\frac{2ab}{a^2+b^2}+2\sqrt{\frac{1}{a}\cdot\frac{1}{b}}\sqrt{2\left(a^2+b^2\right)}=$$

$$=\frac{2ab}{a^2+b^2}+2\sqrt{2}\sqrt{\frac{a^2+b^2}{2ab}}=\frac{2}{t^2}+2\sqrt{2}t\overset{(1)}{\geq}5=RHS$$, unde

$\frac{2}{t^2}+2\sqrt{2}t\overset{(1)}{\geq}5\Leftrightarrow 2\sqrt{2}t^3-5t^2+2\geq 0\Leftrightarrow\left(t-\sqrt{2}\right)\left(2\sqrt{2}t^2-t-\sqrt{2}\right)\geq 0$, care rezultă din $t\geq\sqrt{2}$, vezi $t=\sqrt{\frac{a^2+b^2}{2ab}}\geq\sqrt{2}\Leftrightarrow\frac{a^2+b^2}{2ab}\geq 2\Leftrightarrow(a-b)^2\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b$.

Remarca.

If $a,b>0$ and $\lambda\geq\frac{1}{2}$ then

$$1)\quad\frac{2ab}{a^2+b^2}+\lambda\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2\left(a^2+b^2\right)}\geq 4\lambda+1.$$

Marin Chirciu

Soluție

$$LHS=\frac{2ab}{a^2+b^2}+\lambda\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2\left(a^2+b^2\right)}\overset{AM-GM}{\geq}\frac{2ab}{a^2+b^2}+\lambda\cdot 2\sqrt{\frac{1}{a}\cdot\frac{1}{b}}\sqrt{2\left(a^2+b^2\right)}=$$

$$=\frac{2ab}{a^2+b^2}+\lambda\cdot 2\sqrt{2}\sqrt{\frac{a^2+b^2}{2ab}}=\frac{2}{t^2}+\lambda\cdot 2\sqrt{2}t\overset{(1)}{\geq}1+4\lambda=RHS$$, unde

$$\frac{2}{t^2}+\lambda\cdot 2\sqrt{2}t\overset{(1)}{\geq}1+4\lambda\Leftrightarrow 2\sqrt{2}\lambda t^3-(4\lambda+1)t^2+2\geq 0\Leftrightarrow\left(t-\sqrt{2}\right)\left(2\sqrt{2}\lambda t^2-t-\sqrt{2}\right)\geq 0,$$

care rezultă din $\lambda\geq\frac{1}{2}$ și $t\geq\sqrt{2}$, vezi $t=\sqrt{\frac{a^2+b^2}{2ab}}\geq\sqrt{2}\Leftrightarrow\frac{a^2+b^2}{2ab}\geq 2\Leftrightarrow(a-b)^2\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b$.

Remarca.

If $a,b>0$ then

$$2).\quad\frac{2ab}{a^2+b^2}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2\left(a^2+b^2\right)}\geq 3.$$

Soluție

Cazul $\lambda = \frac{1}{2}$ în inegalitatea de mai sus.

Remarca.

$$3).\ \frac{2ab}{a^2+b^2} + \left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{\frac{a^2+b^2}{2}} \ge 3.$$

Dezvoltări, Marin Chirciu

Soluție

$$LHS = \frac{2ab}{a^2+b^2} + \left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{\frac{a^2+b^2}{2}} \overset{AM-GM}{\ge} \frac{2ab}{a^2+b^2} + 2\sqrt{\frac{1}{a}\cdot\frac{1}{b}}\sqrt{\frac{a^2+b^2}{2}} =$$

$$= \frac{2ab}{a^2+b^2} + \sqrt{2}\sqrt{\frac{a^2+b^2}{ab}} = \frac{2}{t^2} + \sqrt{2}t \overset{(1)}{\ge} 3 = RHS$$, unde

$$\frac{2}{t^2} + \sqrt{2}t \overset{(1)}{\ge} 3 \Leftrightarrow \sqrt{2}t^3 - 3t^2 + 2 \ge 0 \Leftrightarrow \left(t-\sqrt{2}\right)^2\left(\sqrt{2}t+1\right) \ge 0.$$

Egalitatea are loc dacă și numai dacă $a = b$.

Problema333.

If $x, y, z > 0$, $x + y + z = 3$ then

$$\sum x^2 + 15 \ge 6\sum\sqrt{x}.$$

MathAtelier 11/2024, Panagiotis Danousis, Greece

Soluție

FolosimTangent Line Method pentru funcția $f(x) = x^2 - 6\sqrt{x} + 5$ în punctul $x_0 = 1$.

Obținem ecuația tangentei $y = -x + 1$.

Avem $f(x) = x^2 - 6\sqrt{x} + 5 \ge -x + 1$.

$\sum x^2 + 15 \ge 6\sum\sqrt{x} \Leftrightarrow \sum\left(x^2 - 6\sqrt{x} + 5\right) \ge 0$, care rezultă din:

$$\sum\left(x^2 - 6\sqrt{x} + 5\right) \overset{TLM}{\ge} \sum(-x+1) = -\sum x + 3 = -3 + 3 = 0.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

If $x, y, z > 0$, $x + y + z = 3$ and $\lambda \geq 0$ then

$$\sum x^2 + 3\lambda \geq (\lambda + 1)\sum \sqrt{x}.$$

Marin Chirciu

Soluție

FolosimTangent Line Method pentru funcția $f(x) = x^2 - (\lambda+1)\sqrt{x} + \lambda$ în punctul $x_0 = 1$.

Obținem ecuația tangentei $y = \frac{3-\lambda}{2}x + \frac{\lambda-3}{2}$.

Avem $f(x) = x^2 - (\lambda+1)\sqrt{x} + \lambda \geq \frac{3-\lambda}{2}x + \frac{\lambda-3}{2}$.

$\sum x^2 + 3\lambda \geq (\lambda+1)\sum\sqrt{x} \Leftrightarrow \sum\left(x^2 - (\lambda+1)\sqrt{x} + \lambda\right) \geq 0$, care rezultă din:

$$\sum\left(x^2 - (\lambda+1)\sqrt{x} + \lambda\right) \overset{TLM}{\geq} \sum\left(\frac{3-\lambda}{2}x + \frac{\lambda-3}{2}\right) = \frac{3-\lambda}{2}\sum x + 3\frac{\lambda-3}{2} = \frac{3-\lambda}{2}\cdot 3 + 3\frac{\lambda-3}{2} = 0.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

In ΔABC

$$1).\ 3\sum\frac{r^2}{r_a^2} + 5 \geq 2\sum\sqrt{\frac{3r}{r_a}}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$, $x + y + z = 3$ then

$$\sum x^2 + 15 \geq 6\sum\sqrt{x}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \Leftrightarrow \frac{3r}{r_a} + \frac{3r}{r_b} + \frac{3r}{r_c} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obținem:

$$\sum \frac{9r^2}{r_a^2} + 15 \geq 6\sum\sqrt{\frac{3r}{r_a}} \Leftrightarrow 3\sum\frac{r^2}{r_a^2} + 5 \geq 2\sum\sqrt{\frac{3r}{r_a}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2).\ 3\sum\frac{r^2}{h_a^2} + 5 \geq 2\sum\sqrt{\frac{3r}{h_a}}.$$

Dezvoltări, Marin Chirciu

Problema334.

J669.If $a,b,c>0$ then

$$\sum\frac{b+c}{b^2-bc+c^2} + \frac{ab+bc+ca}{abc} \geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}.$$

Mathematical Reflections 4/2024, Nguyen Viet Hung, Vietnam

Soluție

Lema.

If $a,b,c>0$ then

$$\frac{1}{a} + \frac{b+c}{b^2-bc+c^2} \geq \frac{(a+b+c)^2}{a^3+b^3+c^3}.$$

Demonstrație

$$\frac{1}{a} + \frac{b+c}{b^2-bc+c^2} = \frac{a^2}{a^3} + \frac{(b+c)^2}{b^3+c^3} \overset{CS}{\geq} \frac{(a+b+c)^2}{a^3+b^3+c^3}.$$

$$LHS = \sum\frac{b+c}{b^2-bc+c^2} + \frac{ab+bc+ca}{abc} = \sum\left(\frac{1}{a} + \frac{b+c}{b^2-bc+c^2}\right) \overset{Lema}{\geq} \sum\frac{(a+b+c)^2}{a^3+b^3+c^3} =$$

$$= \frac{3(a+b+c)^2}{a^3+b^3+c^3} = RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c$.

Remarca.

If $a,b,c,d>0$ then

$$\sum\frac{a+b}{a^2-ab+b^2} + \frac{abc+bcd+cda+dab}{abcd} \geq \frac{3(a+b+c+d)^2}{a^3+b^3+c^3+d^3}.$$

Marin Chirciu

Soluție

Lema.

If $a,b,c>0$ then

$$\frac{1}{a}+\frac{b+c}{b^2-bc+c^2}\geq\frac{(a+b+c)^2}{a^3+b^3+c^3}.$$

Demonstrație

$$\frac{1}{a}+\frac{b+c}{b^2-bc+c^2}=\frac{a^2}{a^3}+\frac{(b+c)^2}{b^3+c^3}\overset{CS}{\geq}\frac{(a+b+c)^2}{a^3+b^3+c^3}.$$

$$LHS=\sum\frac{a+b}{a^2-ab+b^2}+\frac{abc+bcd+cda+dab}{abcd}=\sum\left(\frac{1}{a}+\frac{b+c}{b^2-bc+c^2}\right)\overset{Lema}{\geq}$$

$$\overset{Lema}{\geq}\sum\frac{(a+b+c)^2}{a^3+b^3+c^3}\overset{CS}{\geq}\frac{\left(\sum(a+b+c)\right)^2}{\sum\left(a^3+b^3+c^3\right)}=\frac{\left(3\sum a\right)^2}{3\sum a^3}=\frac{9\left(\sum a\right)^2}{3\sum a^3}=\frac{3\left(\sum a\right)^2}{\sum a^3}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=d$.

Problema335.

In ΔABC

$$\sum r_a^2\geq\sum m_a^2.$$

RMM 11/2024, Nguyen Hung Cuong, Vietnam

Soluție

Folosind $\sum r_a^2=(4R+r)^2-2p^2$ și $\sum m_a^2=\dfrac{3\left(p^2-r^2-4Rr\right)}{2}$, inegalitatea se scrie:

$(4R+r)^2-2p^2\geq\dfrac{3\left(p^2-r^2-4Rr\right)}{2}\Leftrightarrow 7p^2\leq 32R^2+28Rr+5r^2$, vezi $p^2\leq 4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\left(h_a^2+r_a^2\right)\geq\frac{4r}{R}\sum m_a^2.$$

Marin Chirciu

Soluție

Folosind: $\sum h_a^2 = \frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4R^2}$, $\sum r_a^2 = (4R+r)^2 - 2p^2$ și

$\sum m_a^2 = \frac{3(p^2 - r^2 - 4Rr)}{2}$, inegalitatea se scrie:

$$\frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4R^2} + (4R+r)^2 - 2p^2 \geq \frac{4r}{R} \cdot \frac{3(p^2 - r^2 - 4Rr)}{2} \Leftrightarrow$$

$$\Leftrightarrow p^2(p^2 + 2r^2 - 32Rr - 8R^2) + 64R^4 + 32R^3r + 116R^2r^2 + 32Rr^3 + r^4 \geq 0.$$

Cazul1). Dacă $(p^2 + 2r^2 - 32Rr - 8R^2) \geq 0$, inegalitatea este evidentă.

Cazul2). Dacă $(p^2 + 2r^2 - 32Rr - 8R^2) < 0$, inegalitatea se rescrie:

$64R^4 + 32R^3r + 116R^2r^2 + 32Rr^3 + r^4 \geq p^2(8R^2 + 32Rr - 2r^2 - p^2)$, care rezultă din inegalitatea lui Gerretsen: $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.

Rămâne să arătăm că:

$$64R^4 + 32R^3r + 116R^2r^2 + 32Rr^3 + r^4 \geq (4R^2 + 4Rr + 3r^2)(8R^2 + 32Rr - 2r^2 - 16Rr + 5r^2) \Leftrightarrow$$

$\Leftrightarrow 8R^4 - 16R^3r + 4R^2r^2 - 7Rr^3 - 2r^4 \geq 0 \Leftrightarrow (R - 2r)(8R^3 + 4Rr^2 + r^3) \geq 0$, $R \geq 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema336.

J.2892. In ΔABC

$$\sum \frac{a^2}{4p(p-a)} \geq 1.$$

RMM11/2024,Neculai Stanciu

Soluție

In ΔABC

$$\sum \frac{a^2}{p(p-a)} = \frac{4(R-r)}{r}.$$

$$LHS = \sum \frac{a^2}{4p(p-a)} = \frac{1}{4}\frac{4(R-r)}{r} = \frac{R-r}{r} \overset{Euler}{\geq} 1 = RHS .$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1) \quad 4 \leq \sum \frac{a^2}{p(p-a)} \leq 4\left(\frac{R}{r}-1\right).$$

Marin Chirciu

Soluție

In ΔABC

$$\sum \frac{a^2}{p(p-a)} = \frac{4(R-r)}{r}.$$

Inegalitaea din dreapta.

$$\sum \frac{a^2}{p(p-a)} = \frac{4(R-r)}{r} = 4\left(\frac{R}{r}-1\right).$$

Inegalitaea din stânga.

$$\sum \frac{a^2}{p(p-a)} = \frac{4(R-r)}{r} \overset{Euler}{\geq} 4.$$

$$2) \quad \frac{18R^2}{p} \leq \sum \frac{a^3}{p(p-a)} \leq \frac{8(2R^3-7r^3)}{pr}.$$

Dezvoltări, Marin Chirciu

Soluție

In ΔABC

$$\sum \frac{a^3}{p(p-a)} = \frac{2\left[p^2(2R-3r)+r^2(4R+r)\right]}{pr}.$$

Inegalitaea din dreapta.

$$\sum \frac{a^3}{p(p-a)} = \frac{2\left[p^2(2R-3r)+r^2(4R+r)\right]}{pr} \overset{Gerretsen}{\leq} \frac{8(2R^3-7r^3)}{pr}$$

Inegalitaea din stânga.

$$\sum \frac{a^3}{p(p-a)} = \frac{2\left[p^2(2R-3r)+r^2(4R+r)\right]}{pr} \overset{Gerretsen}{\geq} \frac{18R^2}{p}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema337.

S.2830. In ΔABC

$$\sum \frac{\tan\frac{A}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}} \geq \frac{(4R+r)^2}{p\left(m(4R+r)+3mr\right)}, m,n>0.$$

RMM 11/2024, D.M.Bătinețu-Giurgiu, Neculai Stanciu

Soluție

$$LHS = \sum \frac{\tan\frac{A}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}} = \sum \frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}} = \sum \frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\cdot\frac{r}{p}} \overset{CS}{\geq}$$

$$\overset{CS}{\geq} \frac{\left(\sum\tan\frac{A}{2}\right)^2}{\sum\left(m\tan\frac{A}{2}+n\cdot\frac{r}{p}\right)} = \frac{\left(\frac{4R+r}{p}\right)^2}{m\cdot\frac{4R+r}{p}+3n\cdot\frac{r}{p}} = \frac{(4R+r)^2}{p\left(m(4R+r)+3mr\right)} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ \sum \frac{\cot\frac{A}{2}}{m+n\cot\frac{B}{2}\cot\frac{C}{2}} \geq \frac{p}{r(m+3n)}, m,n>0.$$

Soluție

$$LHS=\sum\frac{\cot\frac{A}{2}}{m+n\cot\frac{B}{2}\cot\frac{C}{2}}=\sum\frac{\cot^2\frac{A}{2}}{m\cot\frac{A}{2}+n\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=\sum\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\cdot\frac{p}{r}}\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\frac{\left(\sum\cot\frac{A}{2}\right)^2}{\sum\left(m\cot\frac{A}{2}+n\cdot\frac{p}{r}\right)}=\frac{\left(\frac{p}{r}\right)^2}{m\cdot\frac{p}{r}+3n\cdot\frac{p}{r}}=\frac{p}{r(m+3n)}=RHS\ .$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2).\ \sum\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}\tan\frac{C}{2}}\geq\frac{6r}{R}\sum\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}}\ .$$

Dezvoltări, Marin Chirciu

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Folosind $\sum\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}\tan\frac{C}{2}}=\frac{(4R+r)^2-2p^2}{pr}$ și $\sum\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}}=\frac{p^2-2r^2-8Rr}{pr}$ avem:

$$\frac{(4R+r)^2-2p^2}{pr}\geq\frac{6r}{R}\cdot\frac{p^2-2r^2-8Rr}{pr}\Leftrightarrow 2p^2(R+3r)\leq 16R^3+8R^2r+49Rr^2+12r^3,$$

care rezultă din inegalitatea lui Gerretsen: $p^2\leq 4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema338.

If $x,y,z>0$, $x^2+y^2+z^2=\frac{3}{4}$ the

$$\frac{1}{(x+y+z)^2}+\frac{1}{xyz(x+y+z)}\geq\frac{52}{9}\ .$$

IneMath 11/2024,Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm $p=x+y+z, q=xy+yz+zx, r=xyz$.

Avem $\frac{3}{4}=x^2+y^2+z^2\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{3}=\frac{p^2}{3}\Rightarrow p\leq\frac{3}{2}$.

$q^2\geq 3pr$, vezi $q^2=\left(xy+yz+zx\right)^2\geq 3xyz\left(x+y+z\right)=3rp$.

$\frac{3}{4}=x^2+y^2+z^2\overset{SOS}{\geq}xy+yz+zx=q\Rightarrow q\leq\frac{3}{4}$.

Rezultă: $\frac{9}{16}\geq q^2\geq 3pr\Rightarrow pr\leq\frac{3}{16}\Rightarrow r\leq\frac{3}{16p}$.

Inegalitatea se scrie:

$\frac{1}{p^2}+\frac{1}{rp}\geq\frac{52}{9}\Leftrightarrow 9\left(p+r\right)\geq 52p^2r\Leftrightarrow 9p\geq r\left(52p^2-9\right)$, care rezultă din $r\leq\frac{3}{16p}$.

Rămâne să arătăm că:

$9p\geq\frac{3}{16p}\left(52p^2-9\right)\Leftrightarrow p^2\leq\frac{9}{4}$, vezi $p\leq\frac{3}{2}$.

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{2}$.

Remarca.

If $x,y,z>0$, $x^2+y^2+z^2=\frac{3}{4}$ and $\lambda,n\geq 0$ then

$$\frac{\lambda}{\left(x+y+z\right)^2}+\frac{n}{xyz\left(x+y+z\right)}\geq\frac{4}{9}\left(\lambda+12n\right).$$

Marin Chirciu

Problema339.

If $x,y,z>0$,solve the system:

$$\begin{cases}x+y+4\sqrt{x+y}=12\\ \frac{x^3}{\left(x+2\right)^2}+\frac{y^3}{\left(y+2\right)^2}=1\end{cases}.$$

MathAtelier 11/2024

Soluție

$$x+y+4\sqrt{x+y}=12 \overset{\sqrt{x+y}=t}{\Leftrightarrow} t^2+4t=12 \Leftrightarrow t^2+4t-12=0 \Leftrightarrow (t-2)(t+6)=0 \Leftrightarrow t=2 \Leftrightarrow$$

$$\Leftrightarrow x+y=4.$$

$$\frac{x^3}{(x+2)^2}+\frac{y^3}{(y+2)^2} \overset{Radon}{\geq} \frac{(x+y)^3}{(x+y+4)^2} \overset{x+y=4}{=} \frac{4^3}{(4+4)^2}=1.$$

Deducem că sistemul admite soloția unică $(x,y)=(2,2)$.

Remarca.

Let $\lambda \geq 0$ fixed. If $x,y,z>0$, solve the system:

$$\begin{cases} x+y+\sqrt{x+y}=20 \\ \dfrac{x^3}{(x+\lambda)^2}+\dfrac{y^3}{(y+\lambda)^2}=\dfrac{1024}{(8+\lambda)^2} \end{cases}.$$

Marin Chirciu

Soluție

$$x+y+\sqrt{x+y}=20 \overset{\sqrt{x+y}=t}{\Leftrightarrow} t^2+t=20 \Leftrightarrow t^2+t-20=0 \Leftrightarrow (t-4)(t+5)=0 \Leftrightarrow t=4 \Leftrightarrow$$

$$\Leftrightarrow x+y=16.$$

$$\frac{x^3}{(x+\lambda)^2}+\frac{y^3}{(y+\lambda)^2} \overset{Radon}{\geq} \frac{(x+y)^3}{(x+y+2\lambda)^2} \overset{x+y=16}{=} \frac{16^3}{(16+2\lambda)^2}=\frac{1024}{(8+\lambda)^2}=1.$$

Deducem că sistemul admite soloția unică $(x,y)=(8,8)$.

Problema340.

JP.564. If $x,y,z>0$, $x^2+y^2+z^2=\dfrac{3}{4}$ then

$$4(x+y+z)+2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 18.$$

RMM 37, Daniel Sitaru, Romania

Remarca.

1).If $x,y,z>0$, $x^2+y^2+z^2=\dfrac{3}{4}$ and $0\leq\lambda\leq 4$ then

$$\lambda\left(x+y+z\right)+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{3}{2}\left(\lambda+4\right).$$

Soluție

Lema.

If $x>0$ and $\lambda\geq 1$ then

$$\lambda x+\frac{1}{x}\geq\left(\lambda-4\right)x+4.$$

Demonstrație

$\lambda x+\frac{1}{x}\geq\left(\lambda-4\right)x+4\Leftrightarrow\left(2x-1\right)^2\geq 0.$

$$LHS=\lambda\sum x+\sum\frac{1}{x}=\sum\left(\lambda x+\frac{1}{x}\right)\overset{Lema}{\geq}\sum\left(\left(\lambda-4\right)x+4\right)=\left(\left(\lambda-4\right)\sum x+12\right)\overset{\sum x\leq\frac{3}{2}}{\geq}\left(\left(\lambda-4\right)\cdot\frac{3}{2}+12\right)=$$

$$=\frac{3}{2}\left(\lambda+4\right)=RHS.$$

Am folosit mai sus $\sum x\leq 3$, care rezultă din: $\frac{3}{4}=x^2+y^2+z^2\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{3}\Leftrightarrow\sum x\leq\frac{3}{2}.$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{2}$.

Problema341.

If $x,y,z>0$, $xy+yz+zx=3xyz$ and $\lambda\geq 1$ then

$$\lambda\left(x+y+z\right)+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3\left(\lambda+1\right).$$

Dezvoltări, Marin Chirciu

Soluție

Lema.

If $x>0$ and $\lambda\geq 1$ then

$$\lambda x+\frac{1}{x}\geq\left(\lambda-1\right)x+2.$$

Demonstrație

$$\lambda x+\frac{1}{x}\geq(\lambda-1)x+2\Leftrightarrow(x-1)^2\geq 0.$$

$$LHS=\lambda\sum x+\sum\frac{1}{x}=\sum\left(\lambda x+\frac{1}{x}\right)\overset{Lema}{\geq}\sum((\lambda-1)x+2)=\left((\lambda-1)\sum x+6\right)\overset{\sum x\geq 3}{\geq}((\lambda-1)\cdot 3+6)=$$
$$=3(\lambda+1)=RHS.$$

Am folosit mai sus $\sum x\geq 3$, care rezultă din:

$$xy+yz+zx=3xyz\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3\Rightarrow 3=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\overset{CS}{\geq}\frac{9}{x+y+z}\Rightarrow x+y+z\geq 3.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Problema341.

J682. If $a,b>0$ then

$$\left(\frac{a}{b}+\frac{b^2}{a^2}\right)\left(\frac{a^2}{b^2}+\frac{b}{a}\right)+\frac{9ab}{a^2-ab+b^2}\geq 13.$$

Mathematical Reflections 6/2024,An Zhenping, China

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^2}\right)\left(t^2+\frac{1}{t}\right)+\frac{9}{t-1+\frac{1}{t}}\geq 13\Leftrightarrow$

$$\Leftrightarrow t^8-t^7+t^6-11t^5+20t^4-11t^3+t^2-t+1\geq 0\Leftrightarrow(t-1)^4\left(t^4+3t^3+7t^2+3t+1\right)\geq 0.$$

Equality occurs if and only if $t=1\Leftrightarrow a=b$.

Remarca.

If $a,b>0$ and $\lambda\leq 9$ then

$$1)\left(\frac{a}{b}+\frac{b^2}{a^2}\right)\left(\frac{a^2}{b^2}+\frac{b}{a}\right)+\frac{\lambda ab}{a^2-ab+b^2}\geq\lambda+4.$$

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^2}\right)\left(t^2+\frac{1}{t}\right)+\frac{\lambda}{t-1+\frac{1}{t}}\geq\lambda+4\Leftrightarrow$

$\Leftrightarrow t^8-t^7+t^6-(\lambda+2)t^5+2(\lambda+1)t^4-(\lambda+2)t^3+t^2-t+1\geq 0 \Leftrightarrow$

$\Leftrightarrow (t-1)^2\left(t^6+t^5+2t^4+(1-\lambda)t^3+2t^2+3t+1\right)\geq 0$, care rezultă din:

$t^6+t^5+2t^4+(1-\lambda)t^3+2t^2+3t+1\geq 0$ pentru $\lambda\leq 9$, vezi:

$t^6+t^5+2t^4+(1-\lambda)t^3+2t^2+t+1\geq 0 \Leftrightarrow t^3+t^2+2t+(1-\lambda)+\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}\geq 0 \Leftrightarrow$

$\Leftrightarrow \left(t^3+\frac{1}{t^3}\right)+\left(t^2+\frac{1}{t^2}\right)+2\left(t+\frac{1}{t}\right)+(1-\lambda)\geq 0 \overset{t+\frac{1}{t}=x\geq 2}{\Leftrightarrow} \left(x^3-3x\right)+\left(x^2-2\right)+2x+(1-\lambda)\geq 0 \Leftrightarrow$

$\Leftrightarrow x^3+x^2-x-1-\lambda\geq 0 \Leftrightarrow x^3+x^2-x-1\geq\lambda$, adevărată din $\lambda\leq 9$.

Este suficient să arătăm că:

$x^3+x^2-x-1\geq 9 \Leftrightarrow x^3+x^2-x-10\geq 0 \Leftrightarrow (x-2)\left(x^2+3x+5\right)\geq 0$, vezi $x=t+\frac{1}{t}\geq 2$.

Equality occurs if and only if $t=1 \Leftrightarrow a=b$.

Remarca.

If $a,b>0$ then

$$2)\left(\frac{a}{b}+\frac{b^3}{a^3}\right)\left(\frac{a^3}{b^3}+\frac{b}{a}\right)+\frac{16ab}{a^2-ab+b^2}\geq 20.$$

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^3}\right)\left(t^3+\frac{1}{t}\right)+\frac{16}{t-1+\frac{1}{t}}\geq 20 \Leftrightarrow$

$\Leftrightarrow t^{10}-t^9+t^8-18t^6+34t^5-18t^4+t^2-t+1\geq 0 \Leftrightarrow$

$\Leftrightarrow (t-1)^4\left(t^6+3t^5+7t^4+14t^3+7t^2+3t+1\right)\geq 0$.

Equality occurs if and only if $t=1 \Leftrightarrow a=b$.

$$3)\left(\frac{a}{b}+\frac{b^4}{a^4}\right)\left(\frac{a^4}{b^4}+\frac{b}{a}\right)+\frac{25ab}{a^2-ab+b^2}\geq 29.$$

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^4}\right)\left(t^4+\frac{1}{t}\right)+\frac{25}{t-1+\frac{1}{t}}\geq 29 \Leftrightarrow$

$\Leftrightarrow t^{12}-t^{11}+t^{10}-27t^7+52t^6-27t^5+34t^5+t^2-t+1\geq 0 \Leftrightarrow$

$\Leftrightarrow (t-1)^4\left(t^8+3t^7+7t^6+14t^5+25t^4+14t^3+7t^2+3t+1\right)\geq 0$.

Equality occurs if and only if $t=1 \Leftrightarrow a=b$.

$$4)\left(\frac{a}{b}+\frac{b^n}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)+\frac{n^2ab}{a^2-ab+b^2}\geq n^2+4, n\in\mathbf{N}.$$

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^n}\right)\left(t^n+\frac{1}{t}\right)+\frac{n^2}{t-1+\frac{1}{t}}\geq n^2+4 \Leftrightarrow$

$\Leftrightarrow t^{2n+4}-t^{2n+3}+t^{2n+2}-\left(n^2+2\right)t^{n+3}+2\left(n^2+1\right)t^{n+2}-\left(n^2+2\right)t^{n+1}+t^2-t+1\geq 0 \Leftrightarrow$

$\Leftrightarrow (t-1)^4\left(t^{2n}+3t^{2n-1}+7t^{2n-2}+...+7t^2+3t+1\right)\geq 0$.

Equality occurs if and only if $t=1 \Leftrightarrow a=b$.

Remarca.

If $a,b>0$ and $n\in\mathbf{N}$ then

$$5)\left(\frac{a}{b}+\frac{b^n}{a^n}\right)\left(\frac{a^n}{b^n}+\frac{b}{a}\right)+\frac{(n+1)^2ab}{a^2-ab+b^2}\geq n^2+2n+5.$$

Dezvoltări, Marin Chirciu

Solution.

With substitution $\frac{a}{b}=t>0$ we have: $\left(t+\frac{1}{t^n}\right)\left(t^n+\frac{1}{t}\right)+\frac{(n+1)^2}{t-1+\frac{1}{t}}\geq n^2+2n+5 \Leftrightarrow$

$\Leftrightarrow t^{2n+4}-t^{2n+3}+t^{2n+2}-\left(n^2+2n+3\right)t^{n+3}+2(n+1)^2t^{n+2}-\left(n^2+2n+3\right)t^{n+1}+t^2-t+1\geq 0 \Leftrightarrow$

$\Leftrightarrow (t-1)^4\left(t^{2n}+3t^{2n-1}+7t^{2n-2}+...+7t^2+3t+1\right)\geq 0$.

Equality occurs if and only if $t=1 \Leftrightarrow a=b$.

Problema342.

O683. If $a,b,c>0$, $a+b+c=1$ then

$$\sum \frac{b+c}{\sqrt{bc+2a^2+2a}} \geq 2.$$

Mathematical Reflections 6/2024, An Zhenping, China

Solution.

Lemma.

If $a,b,c>0$, $a+b+c=1$ then

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} \geq \frac{2(1-a)}{3a+1}.$$

Proof.

Using $bc \leq \frac{(b+c)^2}{4} = \frac{(1-a)^2}{4}$ we have:

$$\frac{b+c}{\sqrt{bc+2a^2+2a}} \geq \frac{1-a}{\sqrt{\frac{(1-a)^2}{4}+2a^2+2a}} = \frac{2(1-a)}{\sqrt{9a^2+6a+1}} = \frac{2(1-a)}{3a+1}.$$

$$LHS = \sum \frac{b+c}{\sqrt{bc+2a^2+2a}} \overset{Lemma}{\geq} \sum \frac{2(1-a)}{3a+1} = \frac{8}{3}\sum \frac{1}{3a+1} - 2 \overset{Titu-Lemma}{\geq} \frac{8}{3}\frac{9}{\sum(3a+1)} - 2 =$$

$$= \frac{8}{3}\cdot\frac{9}{3\sum a+3} - 2 = \frac{8}{3}\cdot\frac{9}{3\cdot 1+3} - 2 = 2 = RHS.$$

Remarcă.

If $a,b,c>0$, $a+b+c=1$ and $\lambda \geq 1$ then

$$\sum \frac{b+c}{\sqrt{bc+\lambda(\lambda-1)a^2+\lambda a}} \geq \frac{6}{\lambda+1}.$$

Marin Chirciu

Solution.

Lemma.

If $a,b,c>0$, $a+b+c=1$ and $\lambda\ge 1$ then

$$\frac{b+c}{\sqrt{bc+\lambda(\lambda-1)a^2+\lambda a}}\ge\frac{2(1-a)}{(2\lambda-1)a+1}.$$

Proof.

Using $bc\le\frac{(b+c)^2}{4}=\frac{(1-a)^2}{4}$ we have:

$$\frac{b+c}{\sqrt{bc+\lambda(\lambda-1)a^2+\lambda a}}\ge\frac{1-a}{\sqrt{\frac{(1-a)^2}{4}+\lambda(\lambda-1)a^2+\lambda a}}=\frac{2(1-a)}{\sqrt{(2\lambda-1)^2a^2+2(2\lambda-1)a+1}}=$$

$$=\frac{2(1-a)}{(2\lambda-1)a+1}.$$

$$LHS=\sum\frac{b+c}{\sqrt{bc+\lambda(\lambda-1)a^2+\lambda a}}\overset{Lemma}{\ge}\sum\frac{2(1-a)}{(2\lambda-1)a+1}=\sum\frac{2\left(\frac{2\lambda}{2\lambda-1}-a-\frac{1}{2\lambda-1}\right)}{(2\lambda-1)a+1}=$$

$$=\frac{4\lambda}{2\lambda-1}\sum\frac{1}{(2\lambda-1)a+1}-\frac{6}{2\lambda-1}\overset{CS}{\ge}\frac{4\lambda}{2\lambda-1}\cdot\frac{9}{(2\lambda-1)\sum a+3}-\frac{6}{2\lambda-1}=$$

$$=\frac{4\lambda}{2\lambda-1}\cdot\frac{9}{(2\lambda-1)\cdot 1+3}-\frac{6}{2\lambda-1}=\frac{4\lambda}{2\lambda-1}\cdot\frac{9}{2(\lambda+1)}-\frac{6}{2\lambda-1}=\frac{6}{\lambda+1}=RHS.$$

Equality occurs if and only if $a=b=c=\frac{1}{3}$.

Problema343.

U682. Evaluate

$$\int\frac{\sin x\cos x}{\sqrt{5+\sin 4x}}dx.$$

Mathematical Reflections 6/2024, Mihaela Berindeanu, București

Solution.

$$I=\int\frac{\sin x\cos x}{\sqrt{5+\sin 4x}}dx=\frac{1}{2}\int\frac{2\sin x\cos x}{\sqrt{5+\sin 4x}}dx=\frac{1}{2}\int\frac{\sin 2x}{\sqrt{5+\sin 4x}}dx=\frac{1}{4}\int\frac{\sin t}{\sqrt{5+\sin 2t}}dt.$$

Lemma.

Prove that

$$\int \frac{\sin t}{\sqrt{5+\sin 2t}}dt = \frac{1}{2}\left(\arcsin\frac{\sin x-\cos x}{\sqrt{6}} - \ln\left(\sin x+\cos x+\sqrt{5+\sin 2x}\right)\right).$$

Proof.

We note $A=\int \frac{\sin t}{\sqrt{5+\sin 2t}}dt$ and $B=\int \frac{\cos t}{\sqrt{5+\sin 2t}}dt$.

We have:

$$A+B=\int \frac{\sin t+\cos t}{\sqrt{5+\sin 2t}}dt=\int \frac{(\cos t-\sin t)'}{\sqrt{5-\left((\cos t-\sin t)^2-1\right)}}dt=\int \frac{v'(t)}{\sqrt{6-v^2(t)}}dt=$$

$$=\arcsin\frac{v(t)}{\sqrt{6}}=\arcsin\frac{\sin t-\cos t}{\sqrt{6}}.$$

$$B-A=\int \frac{\cos t-\sin t}{\sqrt{5+\sin 2t}}dt=\int \frac{(\sin t+\cos t)'}{\sqrt{5+\left((\sin t+\cos t)^2-1\right)}}dt=\int \frac{u'(t)}{\sqrt{4+u^2(t)}}dt=$$

$$=\ln\left(u(t)+\sqrt{4+u^2(t)}\right)=\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right).$$

From: $A+B=\arcsin\frac{\sin t-\cos t}{\sqrt{6}}$ and $B-A=\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right) \Rightarrow$

$$\Rightarrow 2A=\arcsin\frac{\sin t-\cos t}{\sqrt{6}}-\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right) \Rightarrow$$

$$\Rightarrow A=\frac{1}{2}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{6}}-\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right)\right).$$

Using **Lemma** we have:

$$I=\frac{1}{4}\int \frac{\sin t}{\sqrt{5+\sin 2t}}dt=\frac{1}{4}A=\frac{1}{4}\cdot\frac{1}{2}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{6}}-\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right)\right)=$$

$$=\frac{1}{8}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{6}}-\ln\left(\sin t+\cos t+\sqrt{5+\sin 2t}\right)\right).$$

We deduce that $\int \frac{\sin x\cos x}{\sqrt{5+\sin 4x}}dx=\frac{1}{8}\left(\arcsin\frac{\sin 2x-\cos 2x}{\sqrt{6}}-\ln\left(\sin 2x+\cos 2x+\sqrt{5+\sin 4x}\right)\right).$

Remarca.

Evaluate

$$\int \frac{\sin x \cos x}{\sqrt{\lambda + \sin 4x}} dx, \lambda > 1.$$

Marin Chirciu

Solution.

$$I = \int \frac{\sin x \cos x}{\sqrt{\lambda + \sin 4x}} dx = \frac{1}{2} \int \frac{2\sin x \cos x}{\sqrt{\lambda + \sin 4x}} dx = \frac{1}{2} \int \frac{\sin 2x}{\sqrt{\lambda + \sin 4x}} dx = \frac{1}{4} \int \frac{\sin t}{\sqrt{\lambda + \sin 2t}} dt.$$

Lemma.

Prove that

$$\int \frac{\sin t}{\sqrt{5 + \sin 2t}} dt = \frac{1}{2}\left(\arcsin \frac{\sin x - \cos x}{\sqrt{6}} - \ln\left(\sin x + \cos x + \sqrt{5 + \sin 2x} \right) \right).$$

Proof.

We note $A = \int \frac{\sin t}{\sqrt{\lambda + \sin 2t}} dt$ and $B = \int \frac{\cos t}{\sqrt{\lambda + \sin 2t}} dt$.

We have:

$$A + B = \int \frac{\sin t + \cos t}{\sqrt{\lambda + \sin 2t}} dt = \int \frac{(\cos t - \sin t)'}{\sqrt{\lambda - \left((\cos t - \sin t)^2 - 1 \right)}} dt = \int \frac{v'(t)}{\sqrt{\lambda + 1 - v^2(t)}} dt =$$

$$= \arcsin \frac{v(t)}{\sqrt{\lambda + 1}} = \arcsin \frac{\sin t - \cos t}{\sqrt{\lambda + 1}}.$$

$$B - A = \int \frac{\cos t - \sin t}{\sqrt{\lambda + \sin 2t}} dt = \int \frac{(\sin t + \cos t)'}{\sqrt{\lambda + \left((\sin t + \cos t)^2 - 1 \right)}} dt = \int \frac{u'(t)}{\sqrt{\lambda - 1 + u^2(t)}} dt =$$

$$= \ln\left(u(t) + \sqrt{\lambda - 1 + u^2(t)} \right) = \ln\left(\sin t + \cos t + \sqrt{\lambda + \sin 2t} \right).$$

From: $A + B = \arcsin \frac{\sin t - \cos t}{\sqrt{\lambda + 1}}$ and $B - A = \ln\left(\sin t + \cos t + \sqrt{\lambda - 1 + \sin 2t} \right) \Rightarrow$

$$\Rightarrow 2A = \arcsin \frac{\sin t - \cos t}{\sqrt{\lambda + 1}} - \ln\left(\sin t + \cos t + \sqrt{\lambda - 1 + \sin 2t} \right) \Rightarrow$$

$$\Rightarrow A=\frac{1}{2}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{\lambda+1}}-\ln\left(\sin t+\cos t+\sqrt{\lambda-1+\sin 2t}\right)\right).$$

Using **Lemma** we have:

$$I=\frac{1}{4}\int\frac{\sin t}{\sqrt{\lambda+\sin 2t}}dt=\frac{1}{4}A=\frac{1}{4}\cdot\frac{1}{2}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{\lambda+1}}-\ln\left(\sin t+\cos t+\sqrt{\lambda+\sin 2t}\right)\right)=$$

$$=\frac{1}{8}\left(\arcsin\frac{\sin t-\cos t}{\sqrt{\lambda+1}}-\ln\left(\sin t+\cos t+\sqrt{\lambda+\sin 2t}\right)\right).$$

We deduce that $\int\frac{\sin x\cos x}{\sqrt{\lambda+\sin 4x}}dx=\frac{1}{8}\left(\arcsin\frac{\sin 2x-\cos 2x}{\sqrt{\lambda+1}}-\ln\left(\sin 2x+\cos 2x+\sqrt{\lambda+\sin 4x}\right)\right).$

Problema344.

S683. Let OA, OB, OC be rays in the plane such that

$$OB^2=2OA\cdot OC \text{ and } \frac{AB^2}{OA}+\frac{BC^2}{OC}=3\left(OA+OC\right).$$

Find $\sphericalangle AOC$.

Mathematical Reflections 6/2024, Titu Andreescu, USA

Solution .

We note $OA=a, OB=b, OC=c$, $\sphericalangle AOB=x$, $\sphericalangle BOC=y$.

We have: $b^2=2ac$.

Using the cosine theorem we have $AB^2=a^2+b^2-2ab\cos x$ and $BC^2=b^2+c^2-2bc\cos y$.

$$\frac{AB^2}{OA}+\frac{BC^2}{OC}=3\left(OA+OC\right)\Leftrightarrow\frac{a^2+b^2-2ab\cos x}{a}+\frac{b^2+c^2-2bc\cos y}{c}=3\left(a+c\right)\Leftrightarrow$$

$$\Leftrightarrow 2abc\left(\cos x+\cos y\right)=\left(a+c\right)\left(b^2-2bc\right)\overset{b^2=2ac}{\Leftrightarrow}2abc\left(\cos x+\cos y\right)=0\Leftrightarrow$$

$$\Leftrightarrow \cos x+\cos y=0\Leftrightarrow 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=0.$$

Case1). If $\left(OB\subset Int\left(\sphericalangle AOC\right)\right)\Rightarrow\sphericalangle AOC=x+y$.

$$\cos\frac{x+y}{2}=0\Rightarrow\frac{x+y}{2}=90^\circ\Rightarrow x+y=180^\circ\Rightarrow\sphericalangle AOC=180^\circ.$$

Case12. If $\left(OC \subset Int\left(\sphericalangle AOB\right) \Rightarrow \sphericalangle AOC = x - y\right.$.

$$\cos\frac{x-y}{2} = 0 \Rightarrow \frac{x-y}{2} = 90^\circ \Rightarrow x - y = 180^\circ \Rightarrow \sphericalangle AOC = 180^\circ.$$

We deduce $\sphericalangle AOC = 180^\circ$.

Problema345.

J679. Let n be a positive integer. Find a reprezentation of $N = 6^{8n} + 6^{4n+1} + 1$ as the sum of the squares of three positive integers.

Mathematical Reflections 6/2024, Adrian Andreescu, USA

Solution.

$$N = 6^{8n} + 6\cdot 6^{4n} + 1 = \left(6^{8n} - 2\cdot 6^{4n} + 1\right) + 8\cdot 6^{4n} = \left(6^{4n} - 1\right)^2 + 4\cdot 6^{4n} + 4\cdot 6^{4n} =$$

$$= \left(6^{4n} - 1\right)^2 + \left(2\cdot 6^{2n}\right)^2 + \left(2\cdot 6^{2n}\right)^2 = a^2 + b^2 + c^2.$$

Remark.

1). Let n be a positive integer. Find a reprezentation of $N = 6^{8n} + 6^{4n+3} + 1$ as the sum of the squares of three distinct positive integers.

Solution

$$N = 6^{8n} + 6^{4n+3} + 1 = 6^{8n} + 6^3\cdot 6^{4n+3} + 1 = \left(6^{8n} - 2\cdot 6^{4n} + 1\right) + 218\cdot 6^{4n} = \left(6^{4n} - 1\right)^2 + \left(7^2 + 13^2\right)\cdot 6^{4n} =$$
$$= \left(6^{4n} - 1\right)^2 + 7^2\cdot 6^{4n} + 13^2\cdot 6^{4n} = \left(6^{4n} - 1\right)^2 + \left(7\cdot 6^{2n}\right)^2 + \left(13\cdot 6^{2n}\right)^2 = a^2 + b^2 + c^2.$$

2). Arătați că numărul $N = 6^{8n} + 6^{4n+2} + 1, n \in \mathbf{N}$, se poate scrie ca sumă de patru pătrate perfecte.

Soluție.

$$N = 6^{8n} + 6^2\cdot 6^{4n} + 1 = \left(6^{8n} - 2\cdot 6^{4n} + 1\right) + 38\cdot 6^{4n} = \left(6^{4n} - 1\right)^2 + \left(2^2 + 3^2 + 5^2\right)\cdot 6^{4n} =$$

$$= \left(6^{4n} - 1\right)^2 + \left(2^2 + 3^2 + 5^2\right)\cdot 6^{4n} = \left(6^{4n} - 1\right)^2 + \left(2\cdot 6^{2n}\right)^2 + \left(3\cdot 6^{2n}\right)^2 + \left(5\cdot 6^{2n}\right)^2 = a^2 + b^2 + c^2 + d^2.$$

Dezvoltări,Marin Chirciu

Problema346.

JP.556. In acute ΔABC

$$\sum \sin^2 A(\cos B+\cos C)\le 2+\frac{1}{8}\left(13\frac{r}{R}-15\left(\frac{r}{R}\right)^2-6\left(\frac{r}{R}\right)^3\right).$$

RMM-38, Autumn 2025,Marian Ursărescu, Romania

Soluție

Lema.

In ΔABC

$$\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}.$$

$$LHS=\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}\overset{Gerretsen}{\le}\frac{\frac{R(4R+r)^2}{2(2R-r)}(2R-r)-r^2(4R+r)}{4R^3}=$$

$$=\frac{16R^3+8R^2r-7Rr^2-2r^3}{8R^3}=2+\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^3\overset{Euler}{\le}2+\frac{1}{8}\left(13\frac{r}{R}-15\left(\frac{r}{R}\right)^2-6\left(\frac{r}{R}\right)^3\right)=RHS$$

Am fololosit mai sus:

$2+\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^3\overset{Euler}{\le}2+\frac{1}{8}\left(13\frac{r}{R}-15\left(\frac{r}{R}\right)^2-6\left(\frac{r}{R}\right)^3\right)\Leftrightarrow\left(\frac{2r}{R}-1\right)\left(\frac{2r}{R}+5\right)\le 0$, vezi $2r\le R$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ \frac{9}{2}\cdot\frac{r}{R}\le\sum \sin^2 A(\cos B+\cos C)\le 2+\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^3.$$

Marin Chirciu

Soluție

Lema.

In ΔABC

$$\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}.$$

Inegalitatea din dreapta.

$$\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}\overset{Gerretsen}{\le}\frac{\frac{R(4R+r)^2}{2(2R-r)}(2R-r)-r^2(4R+r)}{4R^3}=$$

$$=\frac{16R^3+8R^2r-7Rr^2-2r^3}{8R^3}=2+\frac{r}{R}-\frac{7}{8}\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^3.$$

Inegalitatea din stânga.

$$\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}\overset{Gerretsen}{\ge}\frac{(16Rr-5r^2)(2R-r)-r^2(4R+r)}{4R^3}=$$

$$=\frac{r(32R^2-30Rr+4r^3)}{4R^3}=\frac{r(16R^2-15Rr+2r^3)}{2R^3}\overset{Euler}{\ge}\frac{r\cdot 9R^2}{2R^3}=\frac{9r}{2R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2)\ \frac{p}{2R}\le\sum \cos^2 A(\sin B+\sin C)\le\frac{2p}{R}\left(1-\frac{3r^2}{R^2}\right).$$

Soluție

Lema.

In ΔABC

$$\sum \cos^2 A(\sin B+\sin C)=\frac{p(8R^2+2Rr-r^2-p^2)}{4R^3}.$$

Inegalitatea din dreapta.

$$\sum \cos^2 A(\sin B+\sin C)=\frac{p(8R^2+2Rr-r^2-p^2)}{4R^3}\overset{Gerretsen}{\le}\frac{2p}{R}\left(1-\frac{3r^2}{R^2}\right).$$

Inegalitatea din stânga.

$$\sum \cos^2 A(\sin B+\sin C)=\frac{p(8R^2+2Rr-r^2-p^2)}{4R^3}\overset{Gerretsen}{\ge}\frac{p}{2R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

ΔABC

$$3)\ \sum \sin^2 A(\cos B+\cos C)\le\sqrt{3}\cdot\frac{R}{2r}\sum \cos^2 A(\sin B+\sin C).$$

Dezvoltări, Marin Chirciu

Soluție

Lema1.

In ΔABC

$$\sum \sin^2 A(\cos B+\cos C)=\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}.$$

Lema2.

In ΔABC

$$\sum \cos^2 A(\sin B+\sin C)=\frac{p(8R^2+2Rr-r^2-p^2)}{4R^3}.$$

Folosind **Lemele** inegalitatea se scrie:

$\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}\le\sqrt{3}\cdot\frac{R}{2r}\cdot\frac{p(8R^2+2Rr-r^2-p^2)}{4R^3}$, vezi Mitrinovic $p\ge 3\sqrt{3}r$.

Rămâne să arătăm că:

$\frac{p^2(2R-r)-r^2(4R+r)}{4R^3}\le\sqrt{3}\cdot\frac{R}{2r}\cdot\frac{3\sqrt{3}r(8R^2+2Rr-r^2-p^2)}{4R^3}\Leftrightarrow$

$\Leftrightarrow p^2(13R-2r)\le 72R^3+18R^2r-Rr^2+2r^3$,vezi Gerretsen $p^2\le 4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema347.

JP.557. In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}\le\frac{3}{2}\left(\frac{R}{r}+1\right).$$

RMM-38, Autumn 2025, Marian Ursărescu, Romania

Soluție

Lema.

In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}.$$

$$LHS=\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4Rr}=$$

$$=\frac{4R^2+8Rr+4r^2}{4Rr}=\frac{R^2+2Rr+r^2}{Rr}=2+\frac{r}{R}+\frac{R}{r}\overset{Euler}{\le}\frac{3}{2}\left(\frac{R}{r}+1\right).$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ 5-\frac{r}{R}\le\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}\le 2+\frac{r}{R}+\frac{R}{r}.$$

Marin Chirciu

Soluție

Lema.

In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}.$$

Inegalitatea din dreapta.

$$LHS=\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4Rr}=$$

$$=\frac{4R^2+8Rr+4r^2}{4Rr}=\frac{R^2+2Rr+r^2}{Rr}=2+\frac{r}{R}+\frac{R}{r}.$$

Inegalitatea din stânga.

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2+r^2+4Rr}{4Rr}=\frac{20Rr-4r^2}{4Rr}=5-\frac{r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

$$2)\ 5-\frac{r}{R}\le\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}\le 2+\frac{r}{R}+\frac{R}{r}\le\frac{3}{2}\left(\frac{R}{r}+1\right).$$

Remarca.

$$3)\ 2-\frac{r}{R}\le\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}\le\frac{R}{r}+\frac{r}{R}-1.$$

Soluție

Lema.

In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}=\frac{p^2+r^2-8Rr}{4Rr}.$$

Inegalitatea din dreapta.

$$LHS=\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}=\frac{p^2+r^2-8Rr}{4Rr}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{4Rr}=$$
$$=\frac{4R^2-4Rr+4r^2}{4Rr}=\frac{R^2-Rr+r^2}{Rr}=\frac{R}{r}+\frac{r}{R}-1.$$

Inegalitatea din stânga.

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}=\frac{p^2+r^2-8Rr}{4Rr}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{4Rr}=\frac{8Rr-4r^2}{4Rr}=2-\frac{r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

4) $$3\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}\geq\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}.$$

Dezvoltări, Marin Chirciu

Soluție

Lema1.

In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\sin^2\frac{A}{2}=\frac{p^2+r^2-8Rr}{4Rr}.$$

Lema2.

In ΔABC

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\cos^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{4Rr}.$$

Folosind **Lemele** inegalitatea se scrie:

$3\cdot\frac{p^2+r^2-8Rr}{4Rr}\geq\frac{p^2+r^2+4Rr}{4Rr}\Leftrightarrow p^2\geq 14Rr-r^2$, vezi Gerretsen: $p^2\geq 16Rr-5r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema348.

JP.567. In ΔABC

$$\frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R^2}\leq\sum\sin^3 A\leq\frac{9\sqrt{6}}{8}\sqrt{1-\frac{r}{R}}.$$

RMM-38, Autumn 2025, George Apostolopoulos, Greece

Soluție

Lema.

In ΔABC

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}.$$

Inegalitatea din dreapta.

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}\overset{Gerretsen\&Mitrinovic}{\leq}\frac{3\sqrt{3}R\left(4R^2+4Rr+3r^2-3r^2-6Rr\right)}{8R^3}=$$
$$=\frac{3\sqrt{3}R\left(4R^2-2Rr\right)}{8R^3}=\frac{3\sqrt{3}(2R-r)}{4R}=\frac{3\sqrt{3}}{2}\left(1-\frac{r}{2R}\right)\overset{Euler}{\leq}\frac{9\sqrt{6}}{8}\sqrt{1-\frac{r}{R}}.$$

Am folosit mai sus: $\frac{3\sqrt{3}}{2}\left(1-\frac{r}{2R}\right)\overset{Euler}{\leq}\frac{9\sqrt{6}}{8}\sqrt{1-\frac{r}{R}}\Leftrightarrow R^2-Rr-2r^2\geq 0\Leftrightarrow(R-2r)(R+r)\geq 0$

, vezi $R\geq 2r$,(Euler).

Inegalitatea din stânga.

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}\overset{Gerretsen\&Mitrinovic}{\geq}\frac{2\cdot 3\sqrt{3}r\left(16Rr-5r^2-3r^2-6Rr\right)}{8R^3}=$$
$$=\frac{3\sqrt{3}r\left(10Rr-8r^2\right)}{4R^3}=\frac{3\sqrt{3}r^2}{4R^2}\cdot\frac{(5R-4r)}{R}=\frac{3\sqrt{3}r^2}{4R^2}\left(5-\frac{4r}{R}\right)\overset{Euler}{\geq}\frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R^2}.$$

Am folosit mai sus: $\frac{3\sqrt{3}r^2}{4R^2}\left(5-\frac{4r}{R}\right)\overset{Euler}{\geq}\frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R^2}\Leftrightarrow R\geq 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\ \frac{3\sqrt{3}}{4}\cdot\frac{r^2}{R^2}\left(5-\frac{4r}{R}\right)\le\sum\sin^3 A\le\frac{3\sqrt{3}}{2}\left(1-\frac{r}{2R}\right).$$

Marin Chirciu

Soluție

Lema.

In ΔABC

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}.$$

Inegalitatea din dreapta.

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}\overset{Gerretsen\&Mitrinovic}{\le}\frac{3\sqrt{3}R\left(4R^2+4Rr+3r^2-3r^2-6Rr\right)}{8R^3}=$$

$$=\frac{3\sqrt{3}R\left(4R^2-2Rr\right)}{8R^3}=\frac{3\sqrt{3}\left(2R-r\right)}{4R}=\frac{3\sqrt{3}}{2}\left(1-\frac{r}{2R}\right).$$

Inegalitatea din stânga.

$$\sum\sin^3 A=\frac{2p\left(p^2-3r^2-6Rr\right)}{8R^3}\overset{Gerretsen\&Mitrinovic}{\ge}\frac{2\cdot3\sqrt{3}r\left(16Rr-5r^2-3r^2-6Rr\right)}{8R^3}=$$

$$=\frac{3\sqrt{3}r\left(10Rr-8r^2\right)}{4R^3}=\frac{3\sqrt{3}r^2}{4R^2}\cdot\frac{\left(5R-4r\right)}{R}=\frac{3\sqrt{3}r^2}{4R^2}\left(5-\frac{4r}{R}\right).$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

$$2)\ \frac{9\sqrt{3}}{2}\cdot\frac{r^2}{R^2}\le\frac{3\sqrt{3}r^2}{4R^2}\left(5-\frac{4r}{R}\right)\le\sum\sin^3 A\le\frac{3\sqrt{3}}{2}\left(1-\frac{r}{2R}\right)\le\frac{9\sqrt{6}}{8}\sqrt{1-\frac{r}{R}}.$$

Remarca.

$$3)\ \frac{5}{8}-2\cdot\frac{r^3}{R^3}\le\sum\cos^3 A\le\frac{5}{2}-17\cdot\frac{r^3}{R^3}.$$

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Soluție

Lema.

In ΔABC

$$\sum \cos^3 A = \frac{(2R+r)^3 - 4R^3 - 3rp^2}{4R^3}.$$

Inegalitatea din dreapta.

$$\sum \cos^3 A = \frac{(2R+r)^3 - 4R^3 - 3rp^2}{4R^3} \overset{Gerretsen}{\leq} \frac{(2R+r)^3 - 4R^3 - 3r\left(16Rr - 5r^2\right)}{4R^3} =$$
$$= \frac{4R^3 + 12R^2 r - 42Rr^2 + 16r^3}{4R^3} = \frac{2R^3 + 6R^2 r - 21Rr^2 + 8r^3}{2R^3} \overset{Euler}{\leq} \frac{5R^3 - 34r^3}{2R^3} = \frac{5}{2} - 17 \cdot \frac{r^3}{R^3}.$$

Inegalitatea din stânga.

$$\sum \cos^3 A = \frac{(2R+r)^3 - 4R^3 - 3rp^2}{4R^3} \overset{Gerretsen}{\geq} \frac{(2R+r)^3 - 4R^3 - 3r\left(4R^2 + 4Rr + 3r^2\right)}{4R^3} =$$
$$= \frac{4R^3 - 6Rr^2 - 8r^3}{4R^3} = \frac{2R^3 - 3Rr^2 - 4r^3}{2R^3} \overset{Euler}{\geq} \frac{5R^3 - 16r^3}{8R^3} = \frac{5}{8} - 2 \cdot \frac{r^3}{R^3}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

4) $\sum \sin^3 A \leq 3\sqrt{3} \sum \cos^3 A.$

Dezvoltări,Marin Chirciu

Soluție

Lema1.

In ΔABC

$$\sum \sin^3 A = \frac{2p\left(p^2 - 3r^2 - 6Rr\right)}{8R^3}.$$

Lema2.

In ΔABC

$$\sum \cos^3 A = \frac{(2R+r)^3 - 4R^3 - 3rp^2}{4R^3}.$$

Folosind **Lemele** inegalitatea se scrie:

$$\frac{2p\left(p^2 - 3r^2 - 6Rr\right)}{8R^3} \leq 3\sqrt{3} \cdot \frac{(2R+r)^3 - 4R^3 - 3rp^2}{4R^3} \Leftrightarrow$$

$\Leftrightarrow\ p\left(p^2-3r^2-6Rr\right)\le 3\sqrt{3}\cdot\left[\left(2R+r\right)^3-4R^3-3rp^2\right]$, vezi Mitrinovic $p\le\frac{3\sqrt{3}R}{2}$.

Rămâne să arătăm că:

$$\frac{3\sqrt{3}R}{2}\left(p^2-3r^2-6Rr\right)\le 3\sqrt{3}\cdot\left[\left(2R+r\right)^3-4R^3-3rp^2\right]\Leftrightarrow$$

$\Leftrightarrow\ p^2\left(R+6r\right)\le 8R^3+30R^2r+15Rr^2+2r^3$, vezi Gerretsen: $p^2\le 4R^2+4Rr+3r^2$.

Problema349.

JP.568. In ΔABC

$$4\le\sum\sec^2\frac{A}{2}\le 2\left(\frac{R}{2r}\right)^4+2.$$

RMM-38, Autumn 2025, George Apostolopoulos, Greece

Soluție

Lema.

In ΔABC

$$\sum\sec^2\frac{A}{2}=\frac{p^2+\left(4R+r\right)^2}{p^2}.$$

Inegalitatea din dreapta.

$$\sum\sec^2\frac{A}{2}=\frac{p^2+\left(4R+r\right)^2}{p^2}=1+\frac{\left(4R+r\right)^2}{p^2}\overset{Gerretsen}{\le}1+\frac{\left(4R+r\right)^2}{\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)}}=1+\frac{2\left(2R-r\right)}{R}=$$

$$\frac{5R-2r}{R}=5-\frac{2r}{R}\overset{Euler}{\le}2\left(\frac{R}{2r}\right)^4+2$$

Am folosit mai sus: $5-\frac{2r}{R}\overset{Euler}{\le}2\left(\frac{R}{2r}\right)^4+2\ \Leftrightarrow\ R^5-24Rr^4+16r^5\ge 0\ \Leftrightarrow$

$\left(R-2r\right)\left(R^4+2R^3r+4R^2r^2+8Rr^3-8r^4\right)\ge 0$, vezi $R\ge 2r$,(Euler).

Inegalitatea din stânga.

$$\sum\sec^2\frac{A}{2}=\frac{p^2+\left(4R+r\right)^2}{p^2}=1+\frac{\left(4R+r\right)^2}{p^2}\overset{Doucet}{\ge}4\ \frac{5R-2r}{R}=5-\frac{2r}{R}\overset{Euler}{\le}2\left(\frac{R}{2r}\right)^4+2$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1)\quad 4 \le \sum \sec^2 \frac{A}{2} \le 5 - \frac{2r}{R}.$$

$$2)\quad 4 \le \sum \sec^2 \frac{A}{2} \le 5 - \frac{2r}{R} \le 2\left(\frac{R}{2r}\right)^4 + 2.$$

$$3)\quad 12 \le \sum \csc^2 \frac{A}{2} \le 4\left(\frac{R^2}{r^2} - 1\right).$$

Soluție

Lema.

In ΔABC

$$\sum \csc^2 \frac{A}{2} = \frac{p^2 + r^2 - 8Rr}{r^2}.$$

$$4)\quad 3\sum \sec^2 \frac{A}{2} \le \sum \csc^2 \frac{A}{2}.$$

Dezvoltări,Marin Chirciu

Problema350.

SP.559. In acute ΔABC

$$\frac{r^3}{R} \le \frac{s_a s_b s_c}{a+b+c} \le \frac{Rr}{4}.$$

RMM-38, Autumn 2025, Marian Ursărescu, Romania

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\frac{r^2 p}{R} \le \frac{s_a s_b s_c}{a+b+c} \le \frac{Rp}{4}.$$

Marin Chirciu

Soluție

Lema.

In ΔABC

$$\frac{2r^2p^2}{R} \le s_a s_b s_c \le \frac{Rp^2}{2}.$$

Demonstrație

Folosim: $h_a \le s_a \le m_a \Rightarrow h_a h_b h_c \le s_a s_b s_c \le m_a m_b m_c$

Avem $h_a h_b h_c = \frac{2r^2p^2}{R}$ și $m_a m_b m_c \le \frac{Rp^2}{2}$.

Obținem:

$$\frac{2r^2p^2}{R} \le s_a s_b s_c \le \frac{Rp^2}{2}.$$

Folosind **Lema** obținem:

$$\frac{\frac{2r^2p^2}{R}}{a+b+c} \le \frac{s_a s_b s_c}{a+b+c} \le \frac{\frac{Rp^2}{2}}{a+b+c} \Leftrightarrow \frac{\frac{2r^2p^2}{R}}{2p} \le \frac{s_a s_b s_c}{a+b+c} \le \frac{\frac{Rp^2}{2}}{2p} \Leftrightarrow \frac{r^2p}{R} \le \frac{s_a s_b s_c}{a+b+c} \le \frac{Rp}{4}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema351.

JP.565. If $a,b>0$ then

$$3\sqrt[3]{a}+5\sqrt[5]{a}+3\sqrt[3]{b}+5\sqrt[5]{b} \le 4\sqrt{a}+4\sqrt{b}+8.$$

RMM-38, Autumn 2025, Daniel Sitaru, Romania

Soluție

Lema.

If $a>0$ then

$$3\sqrt[3]{a}+5\sqrt[5]{a} \le 4\sqrt{a}+4.$$

Demonstrație

Notând $\sqrt[30]{a}=t$ inegalitatea $3\sqrt[3]{a}+5\sqrt[5]{a} \le 4\sqrt{a}+4$ se scrie $3t^{10}+5t^6 \le 4t^{15}+4 \Leftrightarrow$

$\Leftrightarrow 4t^{15}-3t^{10}-5t^6+4 \ge 0 \Leftrightarrow$

$(t-1)^2\left(4t^{13}+8t^{12}+12t^{11}+16t^{10}+20t^9+21t^8+22t^7+23t^6+24t^5+20t^4+16t^3+12t^2+8t+4\right) \ge 0$

Să trecem la rezolvarea problemei din enunț.

Folosind **Lema** obținem :

$$LHS=\sum\left(3\sqrt[3]{a}+5\sqrt[5]{a}\right)\overset{Lema}{\leq}\sum\left(4\sqrt{a}+4\right)=4\sqrt{a}+4\sqrt{b}+8=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

Problema se poate dezvolta.

If $a,b,c>0$ then

$$3\sum\sqrt[3]{a}+5\sum\sqrt[5]{a}\leq 4\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+3\right).$$

Marin Chirciu

Soluție

Lema.

If $a>0$ then

$$3\sqrt[3]{a}+5\sqrt[5]{a}\leq 4\sqrt{a}+4.$$

Demonstrație

Notând $\sqrt[30]{a}=t$ inegalitatea $3\sqrt[3]{a}+5\sqrt[5]{a}\leq 4\sqrt{a}+4$ se scrie $3t^{10}+5t^{6}\leq 4t^{15}+4\Leftrightarrow$

$\Leftrightarrow 4t^{15}-3t^{10}-5t^{6}+4\geq 0\Leftrightarrow$

$$(t-1)^2\left(4t^{13}+8t^{12}+12t^{11}+16t^{10}+20t^{9}+21t^{8}+22t^{7}+23t^{6}+24t^{5}+20t^{4}+16t^{3}+12t^{2}+8t+4\right)\geq 0$$

$$LHS=\sum\left(3\sqrt[3]{a}+5\sqrt[5]{a}\right)\overset{Lema}{\leq}\sum\left(4\sqrt{a}+4\right)=4\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+3\right)=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema352.

JP.564. If $x,y,z>0$, $x^2+y^2+z^2=\frac{3}{4}$ then

$$4(x+y+z)+2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 18.$$

RMM-38, Autumn 2025, Daniel Sitaru, Romania

Soluție

Folosim pqr -Method.

Notăm $p = x + y + z, q = xy + yz + zx, r = xyz$.

$$x^2 + y^2 + z^2 = \frac{3}{4} \Leftrightarrow p^2 - 2q = \frac{3}{4} \Leftrightarrow q = \frac{4p^2 - 3}{8}, \text{ (1).}$$

$$\frac{3}{4} = x^2 + y^2 + z^2 \overset{CS}{\geq} \frac{(x+y+z)^2}{3} = \frac{p^2}{3} \Rightarrow \frac{3}{4} \geq \frac{p^2}{3} \Rightarrow p \leq \frac{3}{2}.$$

$q^2 \geq 3rp \Leftrightarrow r \leq \frac{q^2}{3p}$, (2), vezi $q^2 = (xy + yz + zx)^2 \geq 3xyz(x + y + z) = 3rp$.

Inegalitatea de demonstrat se scrie:

$4p + \frac{2q}{r} \geq 18 \Leftrightarrow 2p + \frac{q}{r} \geq 9$,(3).

Din (2) și (3) este suficient să arătăm că:

$$2p + \frac{q}{\frac{q^2}{3p}} \geq 9 \Leftrightarrow 2p + \frac{3p}{q} \geq 9 \overset{(1)}{\Leftrightarrow} 2p + \frac{3p}{\frac{4p^2-3}{8}} \geq 9 \Leftrightarrow 8p^3 - 36p^2 + 18p + 27 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (3p-2)(4p^2 - 12p - 9) \geq 0$, care rezultă din $(3p-2) \geq 0$,vezi $p \leq \frac{3}{2}$ și $4p^2 - 12p - 9 < 0$,vezi $0 < p \leq \frac{3}{2}$, (parabola $x \to 4x^2 - 12x - 9$ are vârful $V\left(\frac{3}{2}, -18\right)$ și pentru $x \in \left(0, \frac{3}{2}\right]$ este negativă).

Egalitatea are loc dacă și numai dacă $x = y = z = \frac{1}{2}$.

Remarca.

If $x, y, z > 0$, $x^2 + y^2 + z^2 = \frac{3}{4}$ and $n \geq \lambda > 0$ then

$$2\lambda(x + y + z) + n\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3(\lambda + 2n).$$

Marin Chirciu

Soluție

Folosim pqr -Method.

Notăm $p = x + y + z, q = xy + yz + zx, r = xyz$.

$x^2+y^2+z^2=\frac{3}{4}\Leftrightarrow p^2-2q=\frac{3}{4}\Leftrightarrow q=\frac{4p^2-3}{8}$, (1).

$\frac{3}{4}=x^2+y^2+z^2\overset{CS}{\geq}\frac{(x+y+z)^2}{3}=\frac{p^2}{3}\Rightarrow\frac{3}{4}\geq\frac{p^2}{3}\Rightarrow p\leq\frac{3}{2}$.

$q^2\geq 3rp\Leftrightarrow r\leq\frac{q^2}{3p}$, (2), vezi $q^2=(xy+yz+zx)^2\geq 3xyz(x+y+z)=3rp$.

Inegalitatea de demonstrat se scrie:

$2\lambda p+n\frac{q}{r}\geq 3(\lambda+2n)$,(3).

Din (2) și (3) este suficient să arătăm că:

$$2\lambda p+n\frac{q}{\frac{q^2}{3p}}\geq 3(\lambda+2n)\Leftrightarrow 2\lambda p+n\frac{3p}{q}\geq 3(\lambda+2n)\overset{(1)}{\Leftrightarrow} 2\lambda p+n\frac{3p}{\frac{4p^2-3}{8}}\geq 3(\lambda+2n)\Leftrightarrow$$

$$8\lambda p^3-12(\lambda+2n)p^2+6p(4n-\lambda)+9(\lambda+2n)\geq 0\Leftrightarrow$$

$\Leftrightarrow (3p-2)\left[4\lambda p^2-12np-3(\lambda+2n)\right]\geq 0$, care rezultă din $(3p-2)\geq 0$,vezi $p\leq\frac{3}{2}$ și

$4\lambda p^2-12np-3(\lambda+2n)$,vezi $0<p\leq\frac{3}{2}$, (parabola $x\to 4\lambda x^2-12nx-3(\lambda+2n)$ are vârful

$V\left(\frac{3n}{2\lambda},-3\left(\lambda^2+\lambda n+3n^2\right)\right)$, $\frac{3n}{2\lambda}\geq\frac{3}{2}$ și pentru $x\in\left(0,\frac{3}{2}\right]$ este negativă).

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{2}$.

Remarca.

If $x,y,z>0$, $xyz\leq 1$ and $n\in\mathbf{N}$ then

$$\sum\left(\frac{1}{x^{n+4}}+\frac{1}{y^{n+2}}+\frac{1}{z^n}\right)\geq\frac{3(x+y+z)}{xyz}.$$

Marin Chirciu

Soluție

$$LHS=\sum\left(\frac{1}{x^{n+4}}+\frac{1}{y^{n+2}}+\frac{1}{z^n}\right)=\sum\left(\frac{1}{x^{n+4}}+\frac{1}{x^{n+2}}+\frac{1}{x^n}\right)\overset{AM-GM}{\geq}\sum 3\sqrt[3]{\frac{1}{x^{n+4}}\cdot\frac{1}{x^{n+2}}\cdot\frac{1}{x^n}}=$$

$$=\sum\frac{3}{x^{n+2}}\overset{(1)}{\geq}\sum\frac{3}{yz}=RHS,$$

unde(1) $\Leftrightarrow \sum\frac{3}{x^{n+2}}\geq\sum\frac{3}{yz}\Leftrightarrow\sum a^{n+2}\geq\sum bc$, cu $(a,b,c)=\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$, $abc=\frac{1}{xyz}\geq 1$.

Avem $\sum a^{n+2}\overset{Chebyshev}{\geq}\frac{1}{3}\sum a^n\sum a^2\overset{AM-GM}{\geq}\sqrt[3]{(abc)^n}\sum a^2\overset{abc\geq 1}{\geq}\sum a^2\overset{SOS}{\geq}\sum bc$.

Egalitatea are loc dacă și numai dacă $a=b=c=1\Leftrightarrow x=y=z=1$.

Problema354.

JP.558. If $a,b,c>0$, $x\geq 0$ then

$$\sum\frac{(a^3+x)(b^3+x)}{ac^2+x}\geq\sum\frac{(ba^2+x)(cb^2+x)}{c^3+x}.$$

RMM-38, Autumn 2025, Daniel Sitaru, Romania

Soluție

Lema.

If $a,b,c>0$, $x\geq 0$ then

$$\frac{(a^3+x)(b^3+x)}{ac^2+x}\geq\frac{(ba^2+x)(cb^2+x)}{c^3+x}.$$

Demonstrație

$$\frac{(a^3+x)(b^3+x)}{ac^2+x}\geq\frac{(ba^2+x)(cb^2+x)}{c^3+x}\Leftrightarrow$$

$$\Leftrightarrow (a^3+x)(b^3+x)(c^3+x)\geq(ba^2+x)(cb^2+x)(ac^2+x)\Leftrightarrow$$

$$\Leftrightarrow a^3b^3c^3+x\sum a^3b^3+x^2\sum a^3+x^3\geq a^3b^3c^3+xabc\sum ab^2+x^2\sum a^2b+x^3\Leftrightarrow$$

$\Leftrightarrow x\sum a^3b^3+x^2\sum a^3\geq xabc\sum ab^2+x^2\sum a^2b$, care rezultă din:

$x\sum a^3b^3\geq xabc\sum ab^2$, vezi $a^3b^3+a^3b^3+b^3c^3\overset{AM-GM}{\geq}3a^2b^3c$ și analoagele și se sumează.

$x^2\sum a^3\geq x^2\sum a^2b$, vezi $a^3+a^3+b^3\overset{AM-GM}{\geq}3a^2b$ și analoagele și se sumează.

Folosind **Lema** și sumând obținem concluzia.

Egalitatea are loc dacă și numai dacă $a=b=c$ sau $x=0$.

Remarca.

If $a,b,c>0$, $x\geq 0$ then

$$1)\sum\frac{\left(a^4+x\right)\left(b^4+x\right)}{ac^3+x}\geq\sum\frac{\left(ba^3+x\right)\left(cb^3+x\right)}{c^4+x}.$$

Soluție

Lema.

If $a,b,c>0$, $x\geq 0$ then

$$\frac{\left(a^4+x\right)\left(b^4+x\right)}{ac^3+x}\geq\frac{\left(ba^3+x\right)\left(cb^3+x\right)}{c^4+x}.$$

Folosind **Lema** și sumând obținem concluzia.

Egalitatea are loc dacă și numai dacă $a=b=c$ sau $x=0$.

Remarca.

If $a,b,c>0$, $x\geq 0$ then

$$2)\sum\frac{\left(a^{n+1}+x\right)\left(b^{n+1}+x\right)}{ac^n+x}\geq\sum\frac{\left(ba^n+x\right)\left(cb^n+x\right)}{c^{n+1}+x},\ n\in\mathbf{N}.$$

Marin Chirciu

Soluție

Lema.

If $a,b,c>0$, $x\geq 0$ then

$$\frac{\left(a^{n+1}+x\right)\left(b^{n+1}+x\right)}{ac^n+x}\geq\frac{\left(ba^n+x\right)\left(cb^n+x\right)}{c^{n+1}+x}.$$

Folosind **Lema** și sumând obținem concluzia.

Egalitatea are loc dacă și numai dacă $a=b=c$ sau $x=0$.

Problema355.

SP.557. Prove the following inequality:

$$\int_{\frac{1}{2024}}^{3} \frac{\sin x}{x} < 2 + \ln 3 .$$

RMM-38, Autumn 2025, Adalbert Kovacs, Romania

Soluție

Folosim $\frac{\sin x}{x} < 1, x > 0$ .Aplicând proprietatea de monotonie a integralei definite obținem:

$$\int_{\frac{1}{2024}}^{3} \frac{\sin x}{x} dx < \int_{\frac{1}{2024}}^{3} dx = x\Big|_{\frac{1}{2024}}^{3} = 3 - \frac{1}{2024} \overset{(1)}{<} 2 + \ln 3 ,$$

unde $3 - \frac{1}{2024} \overset{(1)}{<} 2 + \ln 3 \Leftrightarrow 1 - \frac{1}{2024} < \ln 3 \Leftrightarrow \frac{2023}{2024} < \ln 3$, vezi $\frac{2023}{2024} < 1 = \ln e < \ln 3$.

Remarca.

Problema se poate dezvolta.

Let $n \in \mathbf{N}^*$ fixed. Prove the following inequality:

$$\int_{\frac{1}{n}}^{e} \frac{\sin x}{x} < e .$$

Marin Chirciu

Soluție

Folosim $\frac{\sin x}{x} < 1, x > 0$ .Aplicând proprietatea de monotonie a integralei definite obținem:

$$\int_{\frac{1}{n}}^{e} \frac{\sin x}{x} dx < \int_{\frac{1}{n}}^{e} dx = x\Big|_{\frac{1}{n}}^{e} = e - \frac{1}{n} < e .$$

Problema356.

SP.556. We consider the function $f : (0, \infty) \to \mathbf{R},\ f(x) = \frac{\sqrt{3x + \sin x}}{x}$.

Prove that it is integrable and prove the inequality:

$$\int_{\varepsilon}^{8} f(x) dx < 10 + \ln 2 ,$$

where $1 > \varepsilon > 0$ and $\varepsilon \to 0$.

RMM-38, Autumn 2025, Adalbert Kovacs, Romania

Remarca.

Problema se poate dezvolta.

We consider the function $f:(0,\infty)\to\mathbf{R},\ f(x)=\frac{\sqrt{\lambda x+\sin x}}{x}$.

Prove that it is integrable and prove the inequality:

$$\int_{\varepsilon}^{a} f(x)dx < 2\sqrt{(\lambda+1)a}.$$

Marin Chirciu

where $\lambda\geq 0, a>0$, $1>\varepsilon>0$ and $\varepsilon\to 0$.

Soluție

Folosim Teorema de integrabilitate a funcțiilor continue:

Orice funcție continuă $f:[a,b]\to\mathbf{R}$ este integrabilă pe $[a,b]$.

Cum funcția $f:[\varepsilon,a]\to\mathbf{R}$, $f(x)=\frac{\sqrt{\lambda x+\sin x}}{x}$ este continuă pe $[\varepsilon,a]$ rezultă că este integrabilă pe $[\varepsilon,a]$.

Avem $\sin x<x$, $x>0 \Rightarrow f(x)=\frac{\sqrt{\lambda x+\sin x}}{x}\leq\frac{\sqrt{\lambda x+x}}{x}=\frac{\sqrt{\lambda+1}}{\sqrt{x}}$.

Aplicând proprietatea de monotonie a integralei definite obținem:

$$\int_{\varepsilon}^{a} f(x)dx<\int_{\varepsilon}^{a}\frac{\sqrt{\lambda+1}}{\sqrt{x}}dx=\sqrt{\lambda+1}\cdot 2\sqrt{x}\Big|_{\varepsilon}^{a}=\sqrt{\lambda+1}\left(2\sqrt{a}-2\sqrt{\varepsilon}\right)<2\sqrt{(\lambda+1)a}.$$

Problema357.

JP.566 . In ΔABC

$$8\leq\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{m_a^2}\leq\frac{4R}{r}\left(\frac{R}{r}-1\right).$$

RMM-38, Autumn 2025, Marin Chirciu

Soluție

Inegalitatea din dreapta.

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{m_a^2}\overset{\text{BŁ n d Ł l Ł}}{\le}\frac{R}{r}\sum\frac{a^2}{m_a^2}\overset{m_a^2\ge p(p-a)}{\le}\frac{R}{r}\sum\frac{a^2}{p(p-a)}=\frac{R}{r}\cdot\frac{4(R-r)}{r}=\frac{4R}{r}\left(\frac{R}{r}-1\right).$$

Am folosit mai sus: $\sum\frac{a^2}{m_a^2}\overset{m_a^2\ge p(p-a)}{\le}\sum\frac{a^2}{p(p-a)}=\frac{4(R-r)}{r}=4\left(\frac{R}{r}-1\right)$.

<u>Inegalitatea din stânga.</u>

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{m_a^2}\overset{AM-GM}{\ge}2\sum\frac{a^2}{m_a^2}\overset{(1)}{\ge}2\cdot 4=8.$$

Am folosit mai sus: $\sum\frac{a^2}{m_a^2}\overset{(1)}{\ge}4$, vezi:

$$\sum\frac{a^2}{m_a^2}=\sum\frac{a^2}{\frac{2b^2+2c^2-a^2}{4}}=\sum\frac{4a^2}{2b^2+2c^2-a^2}=\sum\frac{4x}{2y+2z-x}=\sum\frac{4x^2}{2xy+2xz-x^2}\overset{CS}{\ge}$$

$$\overset{CS}{\ge}\frac{4\left(\sum x\right)^2}{\sum\left(2xy+2xz-x^2\right)}=\frac{4\left(\sum x^2+2\sum yz\right)}{4\sum yz-\sum x^2}\overset{(2)}{\ge}4,$$

unde $\frac{4\left(\sum x^2+2\sum yz\right)}{4\sum yz-\sum x^2}\overset{(2)}{\ge}4\Leftrightarrow\sum x^2\ge\sum yz\Leftrightarrow\sum(y-z)^2\ge 0$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

<u>Remarca.</u>

In ΔABC

$$8\le\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{w_a^2}\le\frac{R^6}{8r^6}.$$

Marin Chirciu

<u>Soluție</u>

<u>Inegalitatea din dreapta.</u>

$$\sum\left(\frac{b}{c}+\frac{c}{b}\right)\frac{a^2}{w_a^2}\overset{\text{BŁ n d Ł l Ł}}{\le}\frac{R}{r}\sum\frac{a^2}{w_a^2}\le\frac{R}{r}\sum\frac{a^2}{h_a^2}\overset{(1)}{\le}\frac{R}{r}\cdot\frac{R^5}{8r^5}=\frac{R^6}{8r^6}.$$

Am folosit mai sus: $\sum \frac{a^2}{h_a^2} \overset{(1)}{\leq} \frac{R^5}{8r^5}$, vezi

$$\sum \frac{a^2}{h_a^2} = \sum \frac{a^2}{\left(\frac{2S}{a}\right)^2} = \frac{1}{4S^2} \sum a^4 \overset{(2)}{\leq} \frac{1}{4p^2r^2} \cdot \frac{27R^5}{2r} \overset{Mitrinovic}{\leq} \frac{1}{4 \cdot 27r^2 \cdot r^2} \cdot \frac{27R^5}{2r} = \frac{R^5}{8r^5},$$

unde(2) $\sum a^4 \leq \frac{27R^5}{2r}$, vezi $\sum a^4 = 2\left[p^4 - p^2\left(8Rr + 6r^2\right) + r^2\left(4R + r\right)^2\right]$ și Gerretsen.

Inegalitatea din stânga.

$$\sum \left(\frac{b}{c} + \frac{c}{b}\right) \frac{a^2}{w_a^2} \overset{AM-GM}{\geq} 2 \sum \frac{a^2}{w_a^2} \overset{(2)}{\geq} 2 \cdot 4 = 8.$$

Am folosit mai sus: $\sum \frac{a^2}{w_a^2} \overset{(2)}{\geq} 4$, care rezultă din $\sum \frac{a^2}{w_a^2} \geq \sum \frac{a^2}{m_a^2} \geq 4$,

vezi $\sum \frac{a^2}{m_a^2} = \sum \frac{a^2}{\frac{2b^2 + 2c^2 - a^2}{4}} = \sum \frac{4a^2}{2b^2 + 2c^2 - a^2} = \sum \frac{4x}{2y + 2z - x} = \sum \frac{4x^2}{2xy + 2xz - x^2} \overset{CS}{\geq}$

$$\overset{CS}{\geq} \frac{4\left(\sum x\right)^2}{\sum\left(2xy + 2xz - x^2\right)} = \frac{4\left(\sum x^2 + 2\sum yz\right)}{4\sum yz - \sum x^2} \overset{(2)}{\geq} 4,$$

unde $\frac{4\left(\sum x^2 + 2\sum yz\right)}{4\sum yz - \sum x^2} \overset{(2)}{\geq} 4 \Leftrightarrow \sum x^2 \geq \sum yz \Leftrightarrow \sum (y - z)^2 \geq 0$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema358.

JP.570. In ΔABC

$$\frac{12r^2 p}{R} \leq \sum w_a \sqrt{\frac{b^2 + c^2}{2}} \leq \frac{9R^2}{2}\sqrt{3}.$$

RMM-38, Autumn 2025, Marin Chirciu

Soluție

Inegalitatea din dreapta.

$$\sum w_a \sqrt{\frac{b^2 + c^2}{2}} \overset{CBS}{\leq} \sqrt{\sum w_a^2 \sum \frac{b^2 + c^2}{2}} = \sqrt{\sum w_a^2 \sum a^2} \overset{(1)}{\leq} \sqrt{\frac{27R^2}{4} \cdot 9R^2} = \frac{9R^2}{2}\sqrt{3}.$$

Am folosit mai sus(1): $\sum w_a^2 \le \frac{27R^2}{4}$ și $\sum a^2 \le 9R^2$.

Inegalitatea din stânga.

$$\sum w_a\sqrt{\frac{b^2+c^2}{2}} \overset{CS}{\ge} \sum w_a\sqrt{\frac{(b+c)^2}{4}} = \frac{1}{2}\sum (b+c)w_a \overset{(2)}{\ge} \frac{1}{2}\cdot\frac{24r^2p}{R} = \frac{12r^2p}{R}.$$

Am folosit mai sus: $\sum (b+c)w_a \overset{(2)}{\ge} \frac{24r^2p}{R}$, vezi

$$\sum (b+c)w_a \ge \sum (b+c)h_a = \frac{p\left(p^2+r^2-2Rr\right)}{R} \overset{Gerretsen}{\ge} \frac{24r^2p}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$6S \le \sum aw_a \le \frac{9R^2}{2}\sqrt{3}.$$

Marin Chirciu

Soluție

Inegalitatea din dreapta.

$$\sum aw_a \overset{CBS}{\le} \sqrt{\sum a^2 \sum w_a^2} \overset{(1)}{\le} \sqrt{9R^2\cdot\frac{27R^2}{4}} = \frac{9R^2}{2}\sqrt{3}.$$

Am folosit mai sus(1): $\sum w_a^2 \le \frac{27R^2}{4}$ și $\sum a^2 \le 9R^2$.

Inegalitatea din stânga.

$\sum aw_a \overset{(2)}{\ge} 6S$, vezi $\sum aw_a \ge \sum ah_a = \sum 2S = 6S$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema359.

SP.563. If $a,b,c>0$, $a+b+c=3$ then

$$\sum \frac{2a^4}{3a^2-2a+5} \ge 1.$$

RMM-38, Autumn 2025, Marin Chirciu

Soluție

Lema.

If $a>0$,then

$$\frac{2a^4}{3a^2-2a+5}\geq\frac{10a-7}{9}.$$

Demonstrație

$$\frac{2a^4}{3a^2-2a+5}\geq\frac{10a-7}{9}\Leftrightarrow 18a^4-30a^3+41x^2-64x+35\geq 0\Leftrightarrow (a-1)^2\left(18a^2+6a+35\right)\geq 0.$$

$$LHS=\sum\frac{2a^4}{3a^2-2a+5}\overset{Lema}{\geq}\sum\frac{10a-7}{9}=\frac{10\sum a-21}{9}=\frac{10\cdot 3-21}{9}=1=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema360.

SP.564. If $a,b,c>0$, $(a+b-1)^2=ab$ and $\lambda\geq 0$ then

$$\frac{1}{ab}+\frac{1}{a^2+b^2}+\frac{\lambda\sqrt{ab}}{a+b}\geq 1+\sqrt{\lambda}.$$

RMM-38, Autumn 2025, Marin Chirciu

Soluție

Lema.

If $a,b,c>0$, $(a+b-1)^2=ab$ then

$$a+b\leq 2 \text{ and } ab\leq 1.$$

Demonstrație

$$(a+b-1)^2=ab\leq\frac{(a+b)^2}{4}\Rightarrow (a+b-1)^2\leq\frac{(a+b)^2}{4}\Leftrightarrow |a+b-1|\leq\frac{a+b}{2}\Leftrightarrow$$

$$\Leftrightarrow\frac{-a-b}{2}\leq a+b-1\leq\frac{a+b}{2}\Rightarrow a+b\leq 2.$$

Apoi $2\geq a+b\geq 2\sqrt{ab}\Rightarrow ab\leq 1$.

$$LHS=\frac{1}{ab}+\frac{1}{a^2+b^2}+\frac{\lambda\sqrt{ab}}{a+b}=\frac{1}{2ab}+\left(\frac{1}{2ab}+\frac{1}{a^2+b^2}\right)+\frac{\lambda\sqrt{ab}}{a+b}\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\frac{1}{2ab}+\frac{4}{(a+b)^2}+\frac{\lambda\sqrt{ab}}{a+b}\overset{a+b\leq 2}{\geq}\frac{1}{2ab}+\frac{4}{2^2}+\frac{\lambda\sqrt{ab}}{a+b}=\frac{1}{2ab}+1+\frac{\lambda\sqrt{ab}}{a+b}=$$

$$=1+\left(\frac{1}{2ab}+\frac{\lambda\sqrt{ab}}{a+b}\right)\overset{AM-GM}{\geq}1+2\sqrt{\frac{1}{2ab}\cdot\frac{\lambda\sqrt{ab}}{a+b}}=1+2\sqrt{\frac{1}{2\sqrt{ab}}\cdot\frac{\lambda}{a+b}}\overset{Lema}{\geq}1+2\sqrt{\frac{1}{2\cdot 1}\cdot\frac{\lambda}{2}}=$$

$$=1+\sqrt{\lambda}=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b=\lambda=1$.

Problema361.

SP.564. If $a_1,a_2,...,a_n>0$, $a_1+a_2+...+a_n=n$ then

$$\left(1+\frac{1}{a_1}\right)^{a_2^2}\left(1+\frac{1}{a_2}\right)^{a_3^2}...\left(1+\frac{1}{a_n}\right)^{a_1^2}\geq 2^n.$$

RMM-38, Autumn 2025, Marin Chirciu

Soluție

Fie $A_1=\ln\left(1+\frac{1}{a_1}\right)^{a_2^2}=a_2^2\ln\left(1+\frac{1}{a_1}\right)=a_2^2\int_0^1\frac{dx}{x+a_1}$.

$$\prod\left(1+\frac{1}{a_1}\right)^{a_2^2}=\sum A_1=\sum a_2^2\int_0^1\frac{dx}{x+a_1}=\int_0^1\left(\sum\frac{a_2^2}{x+a_1}\right)dx\overset{CS}{\geq}\int_0^1\frac{\left(\sum a_1\right)^2}{nx+\sum a_1}dx=$$

$$=\int_0^1\frac{n^2}{nx+n}dx=\int_0^1\frac{n}{x+1}dx=n\ln(x+1)\Big|_0^1=n\ln 2.$$

Rezultă $\sum\ln\left(1+\frac{1}{a_1}\right)^{a_2^2}\geq n\ln 2\Leftrightarrow\prod\left(1+\frac{1}{a_1}\right)^{a_2^2}\geq 2^n$.

Egalitatea are loc dacă și numai dacă $a_1=a_2=...=a_n=1$.

Remarca.

In ΔABC

$$1).\ \left(1+\frac{r_a}{3r}\right)^{\left(\frac{3r}{r_b}\right)^2}\left(1+\frac{r_b}{3r}\right)^{\left(\frac{3r}{r_c}\right)^2}\left(1+\frac{r_c}{3r}\right)^{\left(\frac{3r}{r_a}\right)^2}\geq 8.$$

Soluție

Lema.

If $x, y, z > 0, x+y+z=3$ then

$$\left(1+\frac{1}{x}\right)^{y^2}\left(1+\frac{1}{y}\right)^{z^2}\left(1+\frac{1}{z}\right)^{x^2} \geq 8.$$

Soluție

Fie $X = \ln\left(1+\frac{1}{x}\right)^{y^2} = y^2 \ln\left(1+\frac{1}{x}\right) = y^2 \int_0^1 \frac{dt}{t+x}$.

$$\prod\left(1+\frac{1}{x}\right)^{y^2} = \sum X = \sum y^2 \int_0^1 \frac{dt}{t+x} = \int_0^1 \left(\sum \frac{y^2}{t+x}\right) dt \overset{CS}{\geq} \int_0^1 \frac{\left(\sum x\right)^2}{nt+\sum x} dt =$$

$$= \int_0^1 \frac{3^2}{3t+3} dt = \int_0^1 \frac{3}{t+1} dt = 3\ln(t+1)\Big|_0^1 = 3\ln 2.$$

Rezultă $\sum \ln\left(1+\frac{1}{x}\right)^{y^2} \geq 3\ln 2 \Leftrightarrow \prod\left(1+\frac{1}{x}\right)^{y^2} \geq 2^3$.

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Să trecem la rezolvarea problemei din enunț.

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obținem:

$$\left(1+\frac{1}{\frac{3r}{r_a}}\right)^{\left(\frac{3r}{r_b}\right)^2}\left(1+\frac{1}{\frac{3r}{r_a}}\right)^{\left(\frac{3r}{r_c}\right)^2}\left(1+\frac{1}{\frac{3r}{r_a}}\right)^{\left(\frac{3r}{r_a}\right)^2} \geq 8 \Leftrightarrow \left(1+\frac{r_a}{3r}\right)^{\left(\frac{3r}{r_b}\right)^2}\left(1+\frac{r_b}{3r}\right)^{\left(\frac{3r}{r_c}\right)^2}\left(1+\frac{r_c}{3r}\right)^{\left(\frac{3r}{r_a}\right)^2} \geq 8.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

$$2).\ \left(1+\frac{h_a}{3r}\right)^{\left(\frac{3r}{h_b}\right)^2}\left(1+\frac{h_b}{3r}\right)^{\left(\frac{3r}{h_c}\right)^2}\left(1+\frac{h_c}{3r}\right)^{\left(\frac{3r}{h_a}\right)^2} \geq 8.$$

Dezvoltări, Marin Chirciu

Soluție

Lema.

If $x,y,z>0$, $x+y+z=3$ then

$$\left(1+\frac{1}{x}\right)^{y^2}\left(1+\frac{1}{y}\right)^{z^2}\left(1+\frac{1}{z}\right)^{x^2}\geq 8.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{h_a}+\frac{3r}{h_b}+\frac{3r}{h_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{h_a},\frac{3r}{h_b},\frac{3r}{h_c}\right)$ obținem:

$$\left(1+\frac{1}{\frac{3r}{h_a}}\right)^{\left(\frac{3r}{h_b}\right)^2}\left(1+\frac{1}{\frac{3r}{h_a}}\right)^{\left(\frac{3r}{h_c}\right)^2}\left(1+\frac{1}{\frac{3r}{h_a}}\right)^{\left(\frac{3r}{h_a}\right)^2}\geq 8\Leftrightarrow\left(1+\frac{h_a}{3r}\right)^{\left(\frac{3r}{h_b}\right)^2}\left(1+\frac{h_b}{3r}\right)^{\left(\frac{3r}{h_c}\right)^2}\left(1+\frac{h_c}{3r}\right)^{\left(\frac{3r}{h_a}\right)^2}\geq 8.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Bibliografie:

1. C.Năstăsescu, C.Niță, M.Brandiburu, D. Joița , Exerciții de algebră, Editura Didactică și Pedagogică, București, 1992.
2. O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović și P. M. Vasić, „Geometric Inequalities”, Groningen 1969, Olanda .
3. Marin Chirciu, Inegalități trigonometrice, de la inițiere la performanță, Editura Paralela 45, Pitești, 2016.
4. Marin Chirciu, Inegalități cu linii importante în triunghi ,de la inițiere la performanță, Editura Paralela 45, Pitești, 2018.
5. Marin Chirciu, Inegalități cu laturi și cu raze în triunghi, de la inițiere la performanță, Editura Paralela 45, Pitești, 2017.
6. George Apostolopoulos, Greece , Mathematical Inequalites 11/2024.
7. Kostantinos Geronikolas, Greece, Pure Inequalities 11/2024.
8. Socrate Romanidis, Greece, RMM 11/2024.
9. Dang Ngoc Minh, Vietnam, RMM 11/2024.

14. Mehmet Șahin, Turkey, RMM 11/2024.

15.Ertan Yildirim, Turkey, RMM 11/2024.

16.Marian Ursărescu, Roman, RMM-38, Autumn 2025.

15. Nguyen Hung Cuong,Vietnam, RMM 11/2024.
16. Zaza Mzhavanadze, Georgia, RMM 4/2024.

19.Nguyen Minh Tho, Vietnam, Mathematical Inequalites 11/2024.

23.Panagiotis Danousis, Greece, MathAtelier 11/2024.

26. Imad Zak, Lebanon, Mathematical Inequalities 11/2024.

29.Adrian Andreescu, Dallas, USA, Mathematical Reflections, Nr.6/2024.

30.Shirvan Tahirov, Azerbaijan, RMM 11/2024.

31.Elton Papanikola, MathOlymp 11/2024.

32.Daniel Sitaru, Romania, RMM- Number 38, Autumn 2025.

33.Nguyen Viet Hung,Vietnam, Mathematical Reflections 6/2024.

34. Mathew Bailla, MathOlymp 11/2024.

35. Nedyalka Dimitrova, Bulgarian Mathematical Competition-2023.

36. Witcher Ben, Mathematical Inequalities 11/2024.

37.Mihai Vijdeluc, Baia Mare, RMT.

38.Aurel Doboșan, Lugoj, RMT.

39.Eugen Predoiu,Călărași, RMT.

40.Mircea Mario Stoica,Arad, RMT.

41.Radu Ghenghiu, Oradea, RMT.

42.Olimpiada Matematică din Republica Moldova, 1994.

43.Constantin Apostol,Rm.Sărat, Concursul Laurențiu Duican, 1997.

44.Adriana Duță, Făgăraș,Concursul Laurențiu Duican, 1997.

45.Phan Ngoc Chau, Vietnam, RMM 11/2024.

46.Neculai Stanciu, Buzău, RMM 11/2024.

47.An Zhenping, China, Mathematical Reflections, Nr.6/2024.

48.Mihaela Berindeanu, București, Mathematical Reflections, Nr.6/2024.

49.Titu Andreescu, USA, Mathematical Reflections, Nr.6/2024.

50.Adalbert Kovacs, Romania, RMM- Number 38, Autumn 2025.

51.Marin Chirciu, Inegalități algebrice 2,de inițiere la perf, Ed.Paralela 45, Pitești, 2021.

52.Marin Chirciu, Inegalități geometrice2, inițiere și perf, Ed. Paralela 45, Pitești, 2021.

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