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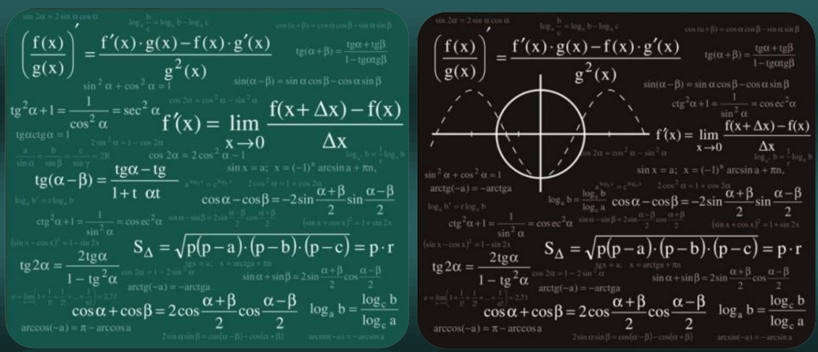

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REDACTORI PRINCIPALI ŞI SUSŢINĂTORI PERMANENŢI AI REVISTEI

NECULAI STANCIU, ROXANA MIHAELA STANCIU ŞI NELA CICEU

Articole :

1.Câteva probleme legate de un triunghi dreptunghic

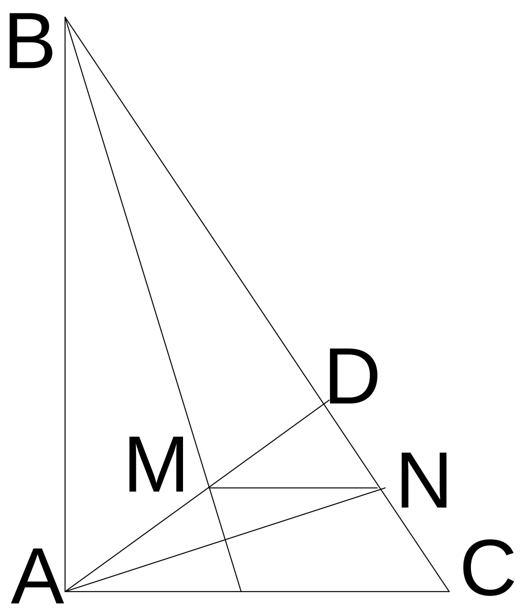
de Roxana Mihaela Stanciu, Buzău

şi Nela Ciceu, Roşiori, Bacău

În cele ce urmează prezentăm câteva probleme (adunate din revistele **Arhimede**, **RMT** şi **Recreaţii Matematice**) care se refera la un triunghi dreptunghic.

1. *În triunghiul* ABC *considerăm simediana* AD, $D \in (BC)$. *Bisectoarea* $\angle ABC$ *intersectează pe* AD *în punctul* M *, iar bisectoarea* $\angle DAC$ *intersectează pe* BC *în punctul* N. *Să se demonstreze ca* $MN \| AC$ *dacă şi numai dacă* $\angle BAC = 90^0$.

<u>Soluţie</u>:



Deoarece, $\frac{BD}{DC} = \frac{c^2}{b^2}$, obţinem $BD = \frac{ac^2}{b^2 + c^2}$.

Se ştie (sau se poate deduce cu relaţia lui *Stewart*) că

$AD = \frac{2bc}{b^2+c^2} \cdot m_a$, unde m_a este mediana din A.

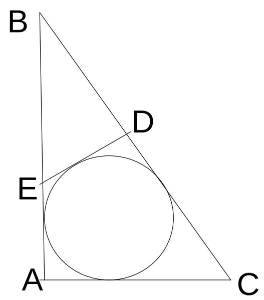
Aplicând teorema bisectoarei, rezultă:

$\frac{MD}{MA} = \frac{BD}{AB}, \frac{ND}{NC} = \frac{AD}{AC}$, şi atunci avem

$$MN \| AC \Leftrightarrow \frac{MD}{MA} = \frac{ND}{NC} \Leftrightarrow \frac{BD}{AB} = \frac{AD}{AC} \Leftrightarrow \frac{abc^2}{b^2+c^2} = \frac{2bc^2 m_a}{b^2+c^2} \Leftrightarrow m_a = \frac{a}{2} \Leftrightarrow A = 90^0.$$

2. *Fie ABC un triunghi dreptunghic în A. Să se construiască cu rigla şi compasul punctele D şi E pe laturile BC, respectiv AB, astfel încât $DE \perp BC$ şi dreapta DE să fie tangentă cercului înscris în triunghiul ABC.*

Soluţie:



Notăm $DC = x$. Din asemănarea triunghiurilor BED şi ABC rezultă

$\frac{DE}{b} = \frac{a-x}{c} = \frac{EB}{a}$, adică

$DE = \frac{b(a-x)}{c}$, şi

$$AE = c - EB = c - \frac{a(a-x)}{c} = \frac{ax + c^2 - a^2}{c} = \frac{ax - b^2}{c}.$$

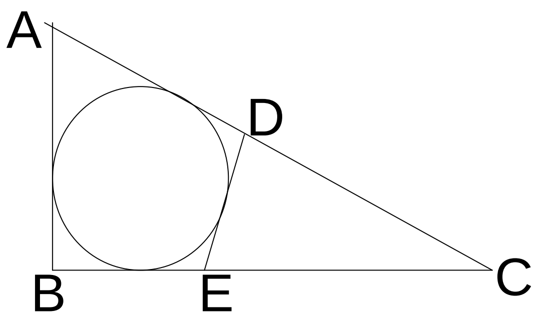
Deoarece patrulaterul $AEDC$ este circumscriptibil, avem

$$AE + DC = DE + AC \Leftrightarrow \frac{ax - b^2}{c} + x = \frac{ab - bx}{c} + b$$

$\Leftrightarrow x(a+b+c) = b^2 + bc + ab \Leftrightarrow x = b$, deci $DC = b$ şi atunci punctul D se construieşte imediat. Perpendiculara în D pe BC intersectează pe AB în punctul căutat E.

3. *Fie ABC un triunghi în care $AB < AC$. Considerăm punctele D şi E pe laturile AC, respectiv BC, astfel încât $AD = AB$ şi $\angle EDC = \angle ABC$.*
Să se demonstreze că dreapta DE este tangentă cercului înscris în triunghiul ABC dacă şi numai dacă $\angle ABC = 90^0$.

Soluţie:



Triunghiurile DEC şi ABC sunt asemenea, deci

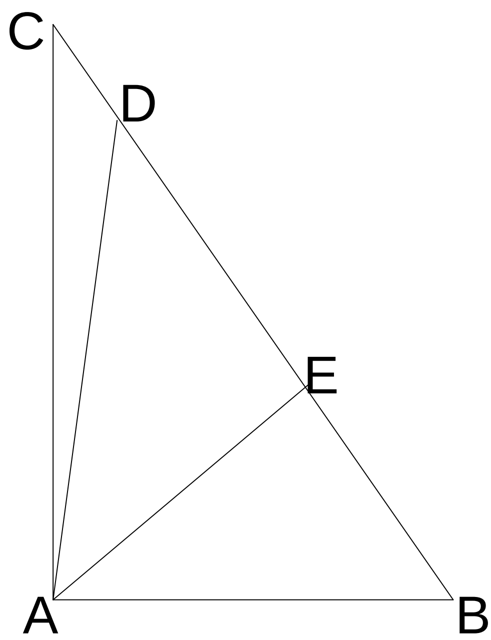
$$\frac{DE}{c} = \frac{DC}{a} = \frac{EC}{b}, \text{ unde } DC = b - c.$$

Dreapta DE este tangentă cercului înscris în triunghiul $ABC \Leftrightarrow ABED$ este circumscriptibil $\Leftrightarrow AB + DE = BE + AD \Leftrightarrow c + \dfrac{c(b-c)}{a} = a - \dfrac{b(b-c)}{a} + c \Leftrightarrow$

$$\Leftrightarrow ac + bc - c^2 = a^2 - b^2 + bc + ac \Leftrightarrow b^2 = a^2 + c^2 \Leftrightarrow \angle B = 90^0.$$

4. *Se consideră triunghiul ABC cu $m(\angle A) = 90^0$ şi $m(\angle B) = 60^0$, iar punctele D şi E de pe latura BC sunt astfel încât AE este bisectoarea unghiului $\angle BAD$, iar $AD = CE$. Determinati măsura unghiului $\angle CAD$.*

Soluţie:



Notăm $m(\angle CAD) = 2x$; folosind teorema sinusurilor în triunghiurile CAD şi CAE, obţinem:

$$AD = \frac{AC \sin 30^0}{\sin(2x + 30^0)}, CE = \frac{AC \sin(x + 45^0)}{\sin(x + 75^0)}.$$

Deoarece $AD = CE$, avem succesiv

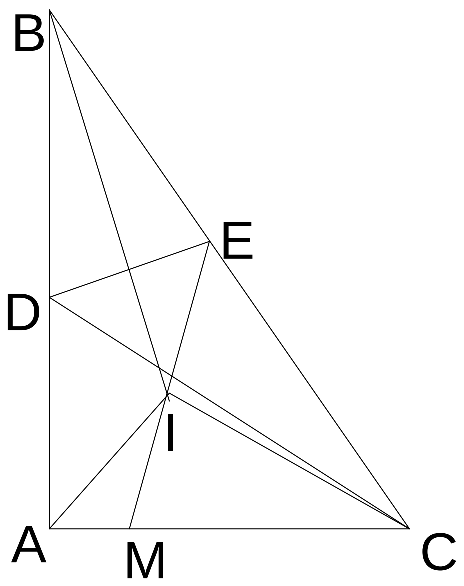
$$\frac{1}{2\sin(2x + 30^0)} = \frac{\sin(x + 45^0)}{\sin(x + 75^0)} \Leftrightarrow 2\sin(2x + 30^0)\sin(x + 45^0) = \sin(x + 75^0)$$

$$\Leftrightarrow \cos(x - 15^0) - \cos(3x + 75^0) = \cos(15^0 - x) \Leftrightarrow \cos(3x + 75^0) = 0,$$

şi cum $2x < 90^0$, rezultă că $3x + 75^0 = 90^0$, adică $m(\angle CAD) = 10^0$.

5. *Fie ABC un triunghi dreptunghic în A şi CD bisectoarea $\angle C$, $D \in (AB)$. Perpendiculara din D pe bisectoarea unghiului B intersectează ipotenuza BC în E. Dacă I este centrul cercului înscris în triunghiul ABC, iar M este punctul de intersecţie dintre EI şi AC, arătaţi că $\angle MIA = \angle IBE$.*

Soluţie:



Din enunţ rezultă că triunghiul BDE este isoscel şi că BI este mediatoarea segmentului DE. Totodată,

$$m(\angle CDE) = m(\angle CDB) - m(\angle EDB) = 180^0 - \frac{C}{2} - B - \frac{180^0 - B}{2} = 45^0 .$$

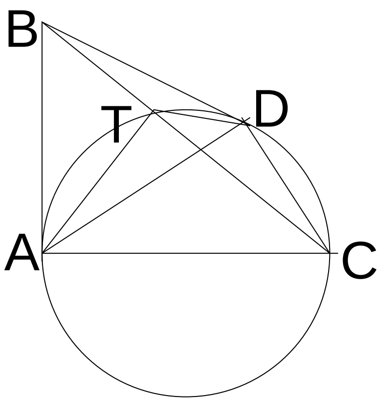
Prin urmare, triunghiul DIE este dreptunghic isoscel. Astfel, $CI \perp ME$ şi cum CI este bisectoarea unghiului C, rezultă că triunghiul CME este isoscel. Deducem că

$$m(\angle MIA) = 180^0 - m(\angle IMA) - 45^0 = 135^0 - 180^0 + m(\angle CME) =$$

$$= -45^0 + \frac{180^0 - C}{2} = \frac{90^0 - C}{2} = \frac{B}{2} = m(\angle IBE) .$$

6. *Fie ABC un triunghi dreptunghic isoscel cu unghiul drept în A. Dacă a doua tangentă dusă din B la cercul de diametru AC intersectează acest cerc în punctul D, să se demonstreze că $AD = 2 \cdot DC$.*

Soluţie:



Fie T punctul de intersecţie al cercului din enunţ cu ipotenuza BC. Deoarece $\angle ATC = 90^0$ (AC este diametru) şi triunghiul ABC este isoscel, rezultă că T este mijlocul lui BC. Aplicând formula medianei în triunghiul DCB, obţinem:

$$DT^2 = \frac{DC^2 + DB^2}{2} - \frac{BC^2}{4} = \frac{DC^2}{2} + \frac{AB^2}{2} - \frac{2 \cdot AB^2}{4} = \frac{DC^2}{2},$$

adică $DC = DT\sqrt{2}$.

Folosind teorema lui *Ptolemeu*, avem:

$$AC \cdot TD + CD \cdot AT = AD \cdot CT \Leftrightarrow AB \cdot TD + CD \cdot AT = AD \cdot CT \Leftrightarrow$$

$$\Leftrightarrow AB \cdot \frac{DC}{\sqrt{2}} + DC \cdot \frac{AB}{\sqrt{2}} = AD \cdot \frac{AB}{\sqrt{2}} \Leftrightarrow 2 \cdot DC = AD.$$

2. Solutions and hints of some problems from the Octogon Mathematical Magazine (II)

By D.M. Bătineţu-Giurgiu, Bucharest, Romania

and Neculai Stanciu, Buzău, Romania

PP. 18898. If $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\sin^8 x + ctg^8 x}{\cos^6 x} + \frac{\cos^8 x + tg^8 x}{\sin^6 x} \geq 8.$

Mihály Bencze

By *AM-GM* inequality and than from *Bergström*'s inequality we have that:

$$\frac{\sin^8 x + ctg^8 x}{\cos^6 x} + \frac{\cos^8 x + tg^8 x}{\sin^6 x} \geq \frac{2\sin^4 x\cos^4 x}{\cos^6 x\sin^4 x} + \frac{2\cos^4 x\sin^4 x}{\sin^6 x\cos^4 x} =$$

$$= \frac{2}{\cos^2 x} + \frac{2}{\sin^2 x} = 2\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) \geq \frac{8}{\sin^2 x + \cos^2 x} = 8$$

and the proof is complete.

PP. 19616. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum_{k=1}^{n} x_k \geq n$ then $\sum_{k=1}^{n} x_k^2 \geq \sum_{k=1}^{n} x_k$.

Mihály Bencze

From the inequality of *Bergström* we deduce that:

$$\sum_{k=1}^{n} x_k^2 \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{n} = \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k\right) \cdot \frac{1}{n},$$

and by

$$\sum_{k=1}^{n} x_k \geq n,$$

we obtain that:

$$\sum_{k=1}^{n} x_k^2 \geq \left(\sum_{k=1}^{n} x_k\right) \cdot n \cdot \frac{1}{n} = \sum_{k=1}^{n} x_k \text{ , q.e.d.}$$

PP. 19844. Prove that if F_n denote the n^{th} Fibonacci number, then

1). $\sum_{k=1}^{n} \frac{F_k^2}{F_{k+1}} \geq \frac{(F_{n+2}-1)^2}{F_{n+3}-2}$

2). $\sum_{k=1}^{n} \frac{F_k^4}{F_{k+1}^2} \geq \frac{(F_n F_{n+1})^2}{F_{n+1}F_{n+2}-1}$

Mihály Bencze

1) By *Bergström*'s inequality we obtain that:

$$\sum_{k=1}^{n} \frac{F_k^2}{F_{k+1}} \geq \frac{\left(\sum_{k=1}^{n} F_k\right)^2}{\sum_{k=1}^{n} F_{k+1}} = \frac{\left(\sum_{k=1}^{n} F_k\right)^2}{\sum_{k=1}^{n+1} F_k - 1} = \frac{(F_{n+2}-1)^2}{F_{n+3}-2}$$, and this is what we

have to prove;

2) Also by *Bergström*'s inequality we obtain that:

$$\sum_{k=1}^{n} \frac{F_k^4}{F_{k+1}^2} = \sum_{k=1}^{n} \frac{\left(F_k^2\right)^2}{F_{k+1}^2} \geq \frac{\left(\sum_{k=1}^{n} F_k^2\right)^2}{\sum_{k=1}^{n} F_{k+1}^2} = \frac{(F_n F_{n+1})^2}{-1+\sum_{k=1}^{n+1} F_k^2} = \frac{(F_n F_{n+1})^2}{F_{n+1}F_{n+2}-1} ,$$

and the proof is complete.

PP. 19901. In all triangle ABC holds $\prod \left(a^2 + 2bc\right) \leq 729R^6$.

Mihály Bencze

Since,

$2xy \leq x^2 + y^2, \forall x, y > 0$ yields that:

$$\prod(a^2+2bc)\le\prod(a^2+b^2+c^2)=(a^2+b^2+c^2)^3,$$

and because:

$$a^2+b^2+c^2\le 9R^2,$$

it follows that:

$$\prod(a^2+2bc)\le(9R^2)^3=729R^6$$, and we are done.

PP. 19659. If $a,b,c,\lambda>0$ then $\sum\limits_{cyclic}\frac{a^\lambda}{b^2-bc+c^2}\le\frac{\sum a^{\lambda+1}}{abc}$.

Mihály Bencze

We have that:

$$\sum_{cyclic}\frac{a^\lambda}{b^2-bc+c^2}\le\sum_{cyclic}\frac{a^\lambda}{2bc-bc}=\sum_{cyclic}\frac{a^\lambda}{bc}=\sum_{cyclic}\frac{a^{\lambda+1}}{abc}=\frac{1}{abc}\sum_{cyclic}a^{\lambda+1}$$, and we are done.

PP. 19931. In all triangle ABC holds $\sum\frac{a^2+r_a^2}{m_a^2}\ge\frac{4\left(2s^2+(4R+r)^2\right)}{3(s^2-r^2-4Rr)}$.

Mihály Bencze

By *Bergström*'s inequality it results that:

$$\sum\frac{a^2+r_a^2}{m_a^2}\ge\frac{\left(\sum a\right)^2+\left(\sum r_a\right)^2}{\sum m_a^2}=\frac{4s^2+(4R+r)^2}{\frac{3}{4}\sum a^2},$$

and because it well-known that:

$$\sum a^2=2(s^2-r^2-4Rr),$$

easy we obtain the conclusion.

PP. 20009. If $x,y,z>0$, then
$x^3+y^3+z^3\ge\frac{3}{2\sqrt{2}}\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\ge 3xyz$.

Mihály Bencze

From direct calculation yields that:

$$x^3+y^3+z^3\ge\frac{3}{2\sqrt{2}}\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)},\forall x,y,z\in R_+^*.$$

Also we have that:

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)\ge 2xy\cdot 2yz\cdot 2zx=8(xyz)^2,\forall x,y,z\in R_+^*.$$

Therefore,

$\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\ge 2\sqrt{2}xyz$, from where we obtain that:

$\frac{3}{2\sqrt{2}}\cdot\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\ge 3xyz$, and the proof is complete.

PP. 20010. If $x_k>0\ (k=1,2,...,n)$, then
$\sum_{k=1}^{n}x_k^3\ge\frac{1}{2\sqrt{2}}\sum_{cyclic}\sqrt{\left(x_1^2+x_2^2\right)\left(x_2^2+x_3^2\right)\left(x_3^2+x_1^2\right)}.$

Mihály Bencze

By the problem 20009 of the Octogon Mathematical Magazine we have that:

$$x^3+y^3+z^3\ge\frac{3}{2\sqrt{2}}\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)},\forall x,y,z\in R_+^*,\text{ so}$$

$$\sum_{cyclic}(x_1^3+x_2^3+x_3^3)\ge\frac{3}{2\sqrt{2}}\sum_{cyclic}\sqrt{(x_1^2+x_2^2)(x_2^2+x_3^2)(x_3^2+x_1^2)}\Leftrightarrow$$

$\Leftrightarrow\sum_{k=1}^{n}x_k^3\ge\frac{1}{2\sqrt{2}}\sum_{cyclic}\sqrt{(x_1^2+x_2^2)(x_2^2+x_3^2)(x_3^2+x_1^2)}$, and the proof is complete.

PP. 20013. In all triangle ABC holds
1). $\sum\sqrt{a}\le\sqrt{\frac{2}{s}(R+r)(4R+r)}$
2). $\sum\sqrt{r_a}\le\sqrt{\frac{(4R+r)^2+s^2}{2R}}$

Mihály Bencze

1)We have that:

$$U=\sum\frac{a}{r_a}=\frac{2(R+r)}{s}=\sum\frac{\left(\sqrt{a}\right)^2}{r_a},$$

where we apply *Bergström's* inequality and follows that:

$$U \geq \frac{\left(\sum\sqrt{a}\right)^2}{\sum r_a} = \frac{\left(\sum\sqrt{a}\right)^2}{4R+r} \Leftrightarrow \frac{2(R+r)}{s} \geq \frac{\left(\sum\sqrt{a}\right)^2}{4R+r} \Leftrightarrow \sum\sqrt{a} \leq \sqrt{\frac{2}{s}(R+r)(4R+r)}\text{, q.e.d.}$$

2) Also by Bergström's inequality we have that:

$$\frac{(4R+r)^2+s^2}{4Rs} = \sum\frac{r_a}{a} = \sum\frac{\left(\sqrt{r_a}\right)^2}{a} \geq \frac{\left(\sum\sqrt{r_a}\right)^2}{\sum a} = \frac{\left(\sum\sqrt{r_a}\right)^2}{2s} \Leftrightarrow$$

$$\Leftrightarrow \sum\sqrt{r_a} \leq \sqrt{\frac{(4R+r)^2+s^2}{2R}}\text{, and the solution is complete.}$$

PP. 20015. In all triangle ABC holds

1). $\sum\sqrt{\frac{a}{r_a}} \leq \sqrt{\frac{6(R+r)}{s}}$

2). $\sum\sqrt{\frac{r_a}{a}} \leq \sqrt{\frac{3\left((4R+r)^2+s^2\right)}{4sR}}$

Mihály Bencze

1)We have that:

$$V = \sum\sqrt{\frac{a}{r_a}} \leq \sqrt{3\cdot\sum\frac{a}{r_a}}\text{, and also we have:}$$

$$\sum\frac{a}{r_a} = \frac{1}{sr}\sum a(s-a) = \frac{1}{sr}\left(s\sum a - \sum a^2\right) = \frac{2(R+r)}{s}.$$

Therefore, we deduce that:

$$V \leq \sqrt{3\cdot\frac{2(R+r)}{s}} = \sqrt{\frac{6(R+r)}{s}}\text{, q.e.d.}$$

$$2)\ W = \sum\sqrt{\frac{r_a}{a}} \leq \sqrt{3\cdot\sum\frac{r_a}{a}} = \sqrt{\frac{3\left((4R+r)^2+s^2\right)}{4Rs}}\text{, and we are done.}$$

PP. 20029. If $a, x \in R$ such that $a + \frac{1}{a} = x$, then $a^9 + \frac{1}{a^9} = \left(x^3 - 3x\right)^3 - 3\left(x^3 - 3x\right)$.

Mihály Bencze

We have that:

$$x^3 = \left(a + \frac{1}{a}\right)^3 = a^3 + \frac{1}{a^3} + 3\left(a + \frac{1}{a}\right) = a^3 + \frac{1}{a^3} + 3x \Rightarrow a^3 + \frac{1}{a^3} = x^3 - 3x.$$

Also we have that:

$$\left(a^3+\frac{1}{a^3}\right)^3=a^9+\frac{1}{a^9}+3\left(a^3+\frac{1}{a^3}\right)\Leftrightarrow$$

$$\Leftrightarrow (x^3-3x)^3=a^9+\frac{1}{a^9}+3(x^3-3x).$$

Hence: $a^9+\frac{1}{a^9}=(x^3-3x)^3-3(x^3-3x)$, q.e.d.

PP. 20040. If $a_i>0$ $(i=1,2,...,n)$ and $k\in\{1,2,...,n\}$, then $\sum\frac{a_1^2}{a_1a_2...a_k(a_2+a_3+...+a_n)^2}\geq\frac{n^2}{(n-1)^2\sum a_1a_2...a_k}$.

Mihály Bencze

Denoting $S_n=\sum_{k=1}^{n}a_k$, we have to prove that:

$$U_n=\sum\frac{a_1^2}{a_1a_2...a_n(a_2+a_3+...+a_n)^2}\geq\frac{n^2}{(n-1)^2\sum a_1a_2...a_n}.$$

Indeed,

$$U_n=\sum\frac{\left(\frac{a_1}{S_n-a_1}\right)^2}{a_1a_2...a_n}$$, and from *Bergström*'s inequality yields that:

$$U_n\geq\frac{\left(\sum\frac{a_1}{S_n-a_1}\right)^2}{\sum a_1a_2...a_n}.$$

By *Nesbitt's* inequality we have that:

$$\sum\frac{a_1}{S_n-a_1}\geq\frac{n}{n-1}.$$

Therefore, we obtain that:

$$U_n\geq\frac{n^2}{(n-1)^2\sum a_1a_2...a_n}$$, and the solution is complete.

PP. 20139. Let n be a nonnegative integer. Show that

$$\frac{F_{n+1}^4+(F_{n+2}^2-F_n^2)^2}{F_{n+1}^2(F_{n+2}^2+F_n^2)}$$

is an integer and determine its value. Here, F_n represents the n^{th} Fibonacci number defined by $F_0=0, F_1=1$ and for all $n\geq 2$, $F_n=F_{n-1}+F_{n-2}$.

José Luis Díaz-Barrero

We have that:

$$\frac{F_{n+1}^4+(F_{n+2}^2-F_n^2)^2}{F_{n+1}^2(F_{n+2}^2+F_n^2)}=\frac{F_{n+1}^4+(F_{n+2}-F_n)^2(F_{n+2}+F_n)^2}{F_{n+1}^2(F_{n+2}^2+F_n^2)}=$$

$$=\frac{F_{n+1}^4+F_{n+1}^2(F_{n+2}+F_n)^2}{F_{n+1}^2(F_{n+2}^2+F_n^2)}=\frac{F_{n+1}^2+(F_{n+2}+F_n)^2}{F_{n+2}^2+F_n^2}=$$

$$=\frac{(F_{n+2}-F_n)^2+(F_{n+2}+F_n)^2}{F_{n+2}^2+F_n^2}=\frac{2(F_{n+2}^2+F_n^2)}{F_{n+2}^2+F_n^2}=2$$, and we are done.

PP. 19258. Let ABC be a triangle and M a random point in his plane. Prove that

1). $\sum\frac{m_a^4MA^4}{2b^2+c^2}\geq\frac{64s^2R^2r^2}{9(s^2-r^2-4Rr)}$

2). $\sum\frac{m_a^4MA^4}{5r_b^2+3r_c^2}\geq\frac{16s^2R^2r^2}{3\left((4R+r)^2-2s^2\right)}$

3). $\sum\frac{m_a^4MA^4}{7\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}}\geq\frac{128s^2R^3r^2}{15(2R-r)}$

4). $\sum\frac{m_a^4MA^4}{5\cos^2\frac{B}{2}+4\cos^2\frac{C}{2}}\geq\frac{256s^2R^3r^2}{27(4R+r)}$

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

1) $$\sum\frac{m_a^4MA^4}{2b^2+c^2}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2MA^2\right)^2}{\sum(2b^2+c^2)}=\frac{\left(\sum m_a^2MA^2\right)^2}{3(a^2+b^2+c^2)}=$$

$$=\frac{\left(\sum m_a^2MA^2\right)^2}{3(s^2-r^2-4Rr)}=\frac{128s^2R^2r^2}{9(s^2-r^2-4Rr)};$$

2) $$\sum\frac{m_a^4MA^4}{5r_b^2+3r_c^2}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2MA^2\right)^2}{\sum(5r_b^2+3r_c^2)}=\frac{\left(\sum m_a^2MA^2\right)^2}{8\sum r_a^2}=$$

$$=\frac{128s^2R^2r^2}{3}\cdot\frac{1}{8\left((4R+r)^2-2s^2\right)}=\frac{16s^2r^2R^2}{3\left((4R+r)^2-2s^2\right)};$$

3) $$\sum\frac{m_a^4MA^4}{7\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2MA^2\right)^2}{\sum\left(7\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}\right)}=\frac{\left(\sum m_a^2MA^2\right)^2}{10\sum\sin^2\frac{A}{2}}=$$

$$=\frac{128s^2R^2r^2}{3}\cdot\frac{1}{10\cdot\frac{2R-r}{2R}}=\frac{128s^2R^3r^2}{15(2R-r)};$$

4) $$\sum\frac{m_a^4MA^4}{5\cos^2\frac{B}{2}+4\cos^2\frac{C}{2}}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2MA^2\right)^2}{\sum\left(5\cos^2\frac{B}{2}+4\cos^2\frac{C}{2}\right)}=\frac{\left(\sum m_a^2MA^2\right)^2}{9\sum\cos^2\frac{A}{2}}=$$

$$=\frac{128s^2R^2r^2}{3}\cdot\frac{1}{9\cdot\frac{4R+r}{2R}}=\frac{256s^2R^3r^2}{27(4R+r)}.$$

The proof is complete.

PP. 19259. If $a_k>0$ $(k=1,2,\dots,n)$, then $\sum_{cyclic}\frac{a_1^{2n-1}}{a_1^{n-1}+a_2a_3\dots a_n}\geq\frac{1}{2}\sum_{k=1}^{n}a_k^n$.

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We have

$$\sum_{cyclic}\frac{a_1^{2n-1}}{a_1^{n-1}+a_2a_3\dots a_n}=\sum_{cyclic}\frac{a_1^{2n}}{a_1^n+a_1a_2a_3\dots a_n}\overset{BERGSTROM}{\geq}\frac{\left(\sum_{cyclic}a_1^n\right)^2}{\sum_{cyclic}a_1^n+n\cdot a_1a_2\dots a_n}\geq$$

$$\geq\frac{\left(\sum_{cyclic}a_1^n\right)^2}{\sum_{k=1}^{n}a_k^n+n\cdot\frac{\sum_{k=1}^{n}a_k^n}{n}}=\frac{\left(\sum_{k=1}^{n}a_k^n\right)^2}{2\sum_{k=1}^{n}a_k^n}=\frac{1}{2}\sum_{k=1}^{n}a_k^n$$, and we are done.

PP. 19260. In all triangle ABC $(a \neq b \neq c)$ holds

1). $\sum \frac{(m_a^2-w_a^2)^2}{(b^2+3c^2)(b-c)^4} \geq \frac{9}{128(s^2-r^2-4Rr)}$

2). $\sum \frac{(m_a^2-w_a^2)^2}{(5r_b^2+2r_c^2)(b-c)^4} \geq \frac{9}{112((4R+r)^2-2s^2)}$

3). $\sum \frac{(m_a^2-w_a^2)^2}{\left(2\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}\right)(b-c)^4} \geq \frac{9R}{40(2R-r)}$

4). $\sum \frac{(m_a^2-w_a^2)^2}{\left(7\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}\right)(b-c)^4} \geq \frac{R}{8(4R+r)}$

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By well-known formulas and *Bergström's* inequality we have

$$1)\sum \frac{(m_a^2-w_a^2)^2}{(b^2+3c^2)(b-c)^4} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{\sum(b^2+3c^2)} = \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{4\sum a^2} =$$

$$= \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{4\cdot 2(s^2-r^2-4Rr)} = \frac{\left(\frac{3}{4}\right)^2}{8(s^2-r^2-4Rr)} = \frac{9}{128(s^2-r^2-4Rr)};$$

$$2)\sum \frac{(m_a^2-w_a^2)^2}{(5r_b^2+2r_c^2)(b-c)^4} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{\sum(5r_b^2+2r_c^2)} = \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{7\sum r_a^2} =$$

$$= \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{7\left((4R+r)^2-2s^2\right)} = \frac{\left(\frac{3}{4}\right)^2}{7\left((4R+r)^2-2s^2\right)} = \frac{9}{112\left((4R+r)^2-2s^2\right)};$$

$$3)\sum \frac{(m_a^2-w_a^2)^2}{\left(2\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}\right)(b-c)^4} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{\sum\left(2\sin^2\frac{B}{2}+3\sin^2\frac{C}{2}\right)} = \frac{\left(\sum \frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{5\sum \sin^2\frac{A}{2}} =$$

$$= \frac{\frac{9}{16}}{5\cdot\frac{2R-r}{2R}} = \frac{9R}{40(2R-r)};$$

$$4)\sum\frac{(m_a^2-w_a^2)^2}{\left(7\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}\right)(b-c)^4}\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{\sum\left(7\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}\right)}=$$

$$=\frac{\left(\sum\frac{m_a^2-w_a^2}{(b-c)^2}\right)^2}{9\sum\cos^2\frac{A}{2}}=\frac{\frac{9}{16}}{9\cdot\frac{4R+r}{2R}}=\frac{R}{8(4R+r)}.$$

The proof is complete.

PP. 19261. Let be M a random point in the plane of triangle ABC. Prove that

1). $\sum\frac{MA^4}{(3a^2+2c^2)h_a^2}\geq\frac{2R^2}{5(s^2-r^2-4Rr)}$

2). $\sum\frac{MA^4}{(5h_b^2+3h_c^2)h_a^2}\geq\frac{2R^4}{(s^2+r^2+4Rr)^2-16s^2Rr}$

3). $\sum\frac{MA^4}{(7r_b^2+2r_c^2)h_a^2}\geq\frac{4R^2}{9s^2}$

4). $\sum\frac{MA^4}{\left(2\sin^2\frac{B}{2}+9\sin^2\frac{C}{2}\right)h_a^2}\geq\frac{8R^3}{11(2R-r)}$

5). $\sum\frac{MA^4}{\left(3\cos^2\frac{B}{2}+7\cos^2\frac{C}{2}\right)h_a^2}\geq\frac{4R^3}{5(4R+r)}$

By well-known formulas and *Bergström's* inequality we have

$$1)\sum\frac{MA^4}{(3b^2+2c^2)h_a^2}\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{MA^2}{h_a}\right)^2}{\sum(3b^2+2c^2)}=\frac{\left(\sum\frac{MA^2}{h_a}\right)^2}{5\sum a^2}=$$

$$=\frac{\left(\sum\frac{MA^2}{h_a}\right)^2}{5\cdot 2(s^2-r^2-4Rr)}\geq\frac{4R^2}{10(s^2-r^2-4Rr)}=\frac{2R^2}{5(s^2-r^2-4Rr)};$$

$$2)\sum\frac{MA^4}{(5h_b^2+3h_c^2)h_a^2}\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{MA^2}{h_a}\right)^2}{\sum(5h_b^2+3h_c^2)}=\frac{\left(\sum\frac{MA^2}{h_a}\right)^2}{8\sum h_a^2}\geq$$

$$\geq\frac{4R^2}{8\sum h_a^2}=\frac{4R^2}{8\cdot\frac{(4Rr+s^2+r^2)^2-16s^2Rr}{4R^2}}=\frac{2R^4}{(s^2+r^2+4Rr)^2-16s^2Rr};$$

$$3)\sum \frac{MA^4}{(7r_b^2+2r_c^2)h_a^2} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{\sum (7r_b^2+2r_c^2)} = \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{9\sum r_a^2} \geq$$

$$\geq \frac{4R^2}{9\sum r_a^2} = \frac{4R^2}{9\left((4R+r)^2-2s^2\right)};$$

$$4)\sum \frac{MA^4}{\left(2\sin^2\frac{B}{2}+9\sin^2\frac{C}{2}\right)h_a^2} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{\sum\left(2\sin^2\frac{B}{2}+9\sin^2\frac{C}{2}\right)} = \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{11\sum \sin^2\frac{A}{2}} \geq$$

$$\geq \frac{4R^2}{11\sum \sin^2\frac{A}{2}} = \frac{4R^2}{11\cdot\frac{2R-r}{2R}} = \frac{8R^3}{11(2R-r)};$$

$$5)\sum \frac{MA^4}{\left(3\cos^2\frac{B}{2}+7\cos^2\frac{C}{2}\right)h_a^2} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{\sum\left(3\cos^2\frac{B}{2}+7\cos^2\frac{C}{2}\right)} = \frac{\left(\sum \frac{MA^2}{h_a}\right)^2}{10\sum \cos^2\frac{A}{2}} \geq$$

$$\geq \frac{4R^2}{10\sum \cos^2\frac{A}{2}} = \frac{2R^2}{5\cdot\frac{4R+r}{2R}} = \frac{4R^3}{5(4R+r)};$$

PP. 19272. In all triangle ABC holds $\sum (b+c)\, m_a^2 \geq \frac{9s\left(s^2-3r^2-4Rr\right)}{4}$.

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

$$\sum (b+c)m_a^2 = \sum \frac{(b+c)^2 m_a^2}{b+c} \overset{BERGSTROM}{\geq} \frac{\left(\sum (b+c)m_a\right)^2}{\sum (b+c)} =$$

$$=\frac{\left(\sum(b+c)m_a\right)^2}{2\sum a}=\frac{9s^2(s^2-3r^2-4Rr)}{4s}=\frac{9s(s^2-3r^2-4Rr)}{4},$$

and we are done.

PP. 19418. If $a_i>0$ $(i=1,2,...,n)$, $k\geq 2$, $\lambda>0$ then $\sum\limits_{cyclic}\frac{a_1\left(a_2+\lambda+\frac{1}{a_2}\right)}{\left(\sqrt[k]{a_2}+\sqrt[k]{a_3}+...+\sqrt[k]{a_n}\right)^k}\geq\frac{(\lambda+2)n}{(n-1)^k}$.

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We use meen inequality, the inequality of *Nesbitt* for n variables and *Radon's* inequality.

$$U=\sum_{cyclic}\frac{a_1\left(a_2+\lambda+\frac{1}{a_2}\right)}{\left(\sqrt[k]{a_2}+\sqrt[k]{a_3}+...+\sqrt[k]{a_n}\right)^k}\overset{AM-GM}{\geq}\sum_{cyclic}\frac{a_1\left(2\cdot\sqrt{a_2\cdot\frac{1}{a_2}}+\lambda\right)}{\left(\sqrt[k]{a_2}+\sqrt[k]{a_3}+...+\sqrt[k]{a_n}\right)^k}=$$

$$=(\lambda+2)\sum_{cyclic}\frac{\left(\sqrt[k]{a_1}\right)^k}{\left(\sqrt[k]{a_2}+\sqrt[k]{a_3}+...+\sqrt[k]{a_n}\right)^k}.$$

Denoting $x_k=\sqrt[k]{a_k}, k=\overline{1,n}$ we have that

$$U\geq(\lambda+2)\sum_{cyclic}\left(\frac{x_1}{S_n-x_1}\right)^k,$$

where we denote $S_n=\sum\limits_{k=1}^{n}x_k$.

By *Radon*'s inequality we obtain

$$U\geq(\lambda+2)\cdot\frac{\left(\sum\limits_{k=1}^{n}\frac{x_k}{S_n-x_k}\right)^k}{n^{k-1}},$$

and then by *Nesbitt's* inequality we deduce that

$$U\geq(\lambda+2)\cdot\frac{1}{n^{k-1}}\cdot\left(\frac{n}{n-1}\right)^k=\frac{(\lambda+2)n}{(n-1)^k},$$

and this completes the proof.

PP. 19443. If $x_k > 0$ $(k = 1, 2, ..., n)$, then $\sqrt{n} + \sum_{k=1}^{n} x_k^2 > \sum_{cyclic} x_1\sqrt{1 + x_2^2}$.

Mihály Bencze

We have

$$\frac{n}{2} + \sum_{k=1}^{n} x_k^2 = \frac{n + 2\sum_{k=1}^{n} x_k^2}{2} = \frac{\sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n}(1 + x_{k+1}^2)}{2} = \frac{1}{2}\sum_{k=1}^{n}(x_k^2 + 1 + x_{k+1}^2) \geq$$

$$\geq \frac{1}{2}\sum_{k=1}^{n} 2\sqrt{x_k^2(1 + x_{k+1}^2)} = \sum_{k=1}^{n} x_k\sqrt{1 + x_{k+1}^2} = \sum_{cyclic} x_1\sqrt{1 + x_2^2}.$$

We note that the inequality is strictly because $x_k^2 \neq 1 + x_{k+1}^2, \ \forall k = \overline{1, n}$.

The proof is complete.

PP. 19495. Let ABC be a triangle. Determine all $\lambda \in R$ for which $\prod \sin \lambda A \geq \frac{s^2 - (2R + r)^2}{4R^2}$.

Mihály Bencze

We have

$$\sum \frac{a^2}{m_b m_c} \overset{BERGSTROM}{\geq} \frac{(a + b + c)^2}{m_a m_b + m_b m_c + m_c m_a} \geq \frac{4s^2}{m_a^2 + m_b^2 + m_c^2} =$$

$$= \frac{4s^2}{\frac{3}{4}(a^2 + b^2 + c^2)} = \frac{16s^2}{3} \cdot \frac{1}{a^2 + b^2 + c^2} = \frac{16s^2}{3} \cdot \frac{1}{2(s^2 - r^2 - 4Rr)} =$$

$$= \frac{8s^2}{3(s^2 - r^2 - 4Rr)} =$$, and we are done.

PP. 19431. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \frac{1}{a_1^{n-1} + a_2^{n-1} + ... + a_{n-1}^{n-1}} \le \frac{\sum_{k=1}^{n} a_k}{(n-1)\prod_{k=1}^{n} a_k}.$$

Mihály Bencze

We have

$$\sum_{cyclic} \frac{1}{a_1^{n-1} + a_2^{n-1} + ... + a_{n-1}^{n-1}} \le \sum_{cyclic} \frac{1}{(n-1)\cdot \sqrt[n-1]{(a_1 a_2 ... a_{n-1})^{n-1}}} =$$

$$= \frac{1}{n-1}\sum_{cyclic} \frac{1}{a_1 a_2 ... a_{n-1}} = \frac{1}{n-1}\sum_{cyclic} \frac{a_n}{a_1 a_2 ... a_n} = \frac{\sum_{k=1}^{n} a_k}{(n-1)\prod_{k=1}^{n} a_k},$$

and we are done.

PP. 19614. In all triangle ABC holds $\sum \left(\frac{a}{m_b}\right)^2 \ge \frac{8s^2}{3(s^2 - r^2 - 4Rr)}$.

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

$$\sum \left(\frac{a}{m_b}\right)^2 \overset{BERGSTROM}{\ge} \frac{\left(\sum a\right)^2}{\sum m_b^2} = \frac{4s^2}{\frac{3}{4}\sum a^2} = \frac{8s^2}{3(s^2 - r^2 - 4Rr)}.$$

The proof is complete.

PP. 19617. In all triangle ABC holds

1). $\sum \frac{\sin^4 \frac{A}{2}}{m_b} \ge \frac{(2R - r)^2}{4R^2(4R + r)}$

2). $\sum \frac{\cos^4 \frac{A}{2}}{m_b} \ge \frac{4R + r}{4R^2}$

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

1) $$\sum\frac{\sin^4\frac{A}{2}}{m_b}\overset{BERGSTROM}{\geq}\frac{\left(\sum\sin^2\frac{A}{2}\right)^2}{\sum m_b}=\left(\frac{2R-r}{2R}\right)^2\cdot\frac{1}{\sum m_a}\geq\frac{(2R-r)^2}{4R^2(4R+r)};$$

2) $$\sum\frac{\cos^4\frac{A}{2}}{m_b}\overset{BERGSTROM}{\geq}\frac{\left(\sum\cos^2\frac{A}{2}\right)^2}{\sum m_b}=\left(\frac{4R+r}{2R}\right)^2\cdot\frac{1}{\sum m_a}\geq\frac{(4R+r)^2}{4R^2(4R+r)}=\frac{4R+r}{4R^2}.$$

The proof is complete.

PP. 19619. In all triangle ABC holds $\sum\frac{m_a^4}{w_bw_c}\geq\frac{9\left(s^2-r^2-4Rr\right)^2}{4s^2}$.

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

$$\sum\frac{m_a^4}{w_bw_c}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2\right)^2}{\sum w_aw_b}=\frac{9}{16}\cdot\frac{\left(\sum a^2\right)^2}{\sum w_aw_b}=\frac{9}{16}\cdot\frac{4(s^2-r^2-4Rr)^2}{\sum w_aw_b}=$$

$$=\frac{9}{4}\cdot\frac{(s^2-r^2-4Rr)^2}{\sum w_aw_b}\geq\frac{9(s^2-r^2-4Rr)^2}{4s^2},$$ and we are done.

PP. 19620. In all triangle ABC holds $\sum\frac{m_a^4}{w_b}\geq\frac{\left(s^2-r^2-4Rr\right)^2}{4r}$.

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

$$\sum\frac{m_a^4}{w_b}\overset{BERGSTROM}{\geq}\frac{\left(\sum m_a^2\right)^2}{\sum w_a}=\frac{9}{16}\cdot\frac{\left(\sum a^2\right)^2}{\sum w_a}=\frac{9}{16}\cdot\frac{4(s^2-r^2-4Rr)^2}{\sum w_a}=$$

$$=\frac{9}{4}\cdot\frac{(s^2-r^2-4Rr)^2}{\sum w_a}\geq\frac{(s^2-r^2-4Rr)^2}{4r},$$ and we are done.

PP. 19621. In all triangle ABC holds $\sum\frac{m_a^2}{m_b}\geq 9r$.

Mihály Bencze

By well-known formulas and Bergström's inequality we have

$$\sum \frac{m_a^2}{m_b} \overset{BERGSTROM}{\geq} \frac{\left(\sum m_a\right)^2}{\sum m_a} = \sum m_a \geq 9r \text{, and we are done.}$$

PP. 19622. In all triangle ABC holds $\sum \frac{a^2}{m_b} \geq \frac{4s^2}{4R+r}$.

Mihály Bencze

By well-known formulas and Bergström's inequality we have

$$\sum \frac{a^2}{m_b} \overset{BERGSTROM}{\geq} \frac{\left(\sum a\right)^2}{\sum m_a} = \frac{4s^2}{m_a + m_b + m_c} \geq \frac{4s^2}{4R+r} \text{, and we are done.}$$

PP. 19623. In all triangle ABC holds $\sum \frac{r_a^2}{m_b} \geq 4R + r$.

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

$$\sum \frac{r_a^2}{m_b} \overset{BERGSTROM}{\geq} \frac{\left(\sum r_a\right)^2}{\sum m_a} \geq \frac{(4R+r)^2}{4R+r} = 4R + r \text{, and we are done.}$$

PP. 19625. In all triangle ABC holds $\sum \frac{m_a^4}{b+c} \geq \frac{9\left(s^2 - r^2 - 4Rr\right)^2}{16s}$.

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

$$\sum \frac{m_a^4}{b+c} \overset{BERGSTROM}{\geq} \frac{\left(\sum m_a^2\right)^2}{\sum (b+c)} = \frac{4}{9} \cdot \frac{\left(\sum a^2\right)^2}{2\sum a} = \frac{2}{9} \cdot \frac{\left(\sum a^2\right)^2}{2s} =$$

$$= \frac{1}{9s} \cdot 4(s^2 - r^2 - 4Rr)^2 = \frac{4}{9s}(s^2 - r^2 - 4Rr)^2 \text{, q.e.d.}$$

PP. 19626. In all triangle ABC holds $\sum \frac{m_a^4}{m_b m_c} \geq \frac{3}{2}\left(s^2 - r^2 - 4Rr\right)$.

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

$$\sum \frac{m_a^4}{m_b m_c} \overset{BERGSTROM}{\geq} \frac{\left(\sum m_a^2\right)^2}{\sum m_a m_b} \geq \sum m_a^2 = \frac{3}{4}\sum a^2 = \frac{3}{4}\cdot 2(s^2 - r^2 - 4Rr) =$$

$$= \frac{3}{2}(s^2 - r^2 - 4Rr)\text{, and we are done.}$$

PP. 19628. In all triangle ABC holds $\sum \frac{w_a^2}{bc} = \frac{(s^2+r^2)+4s^2r(4R+3r)+8Rr^3}{(s^2+r^2+2Rr)^2}$.

Mihály Bencze

From some algebra and well-known formulas yields that

$$3 - \sum \frac{w_a^2}{bc} = \sum \left(1 - \frac{w_a^2}{bc}\right) = \sum \frac{bc - w_a^2}{bc} = \sum \left(\frac{a}{b+c}\right)^2 =$$

$$= \left(\sum \frac{a}{b+c}\right)^2 - 2\cdot \sum \frac{ab}{(b+c)(c+a)} = \frac{(s^2+r^2)^2 + 4s^2r(4R+3r) + 8Rr^3}{(s^2+r^2+2Rr)^2}\text{, q.e.d}$$

PP. 19629. In all triangle ABC holds $\sum \frac{w_a^2}{bc} \leq \frac{9}{4}$.

Mihály Bencze

We use *Bergström*'s inequality and *Nesbitt*'s inequality.

$$3 - \sum \frac{w_a^2}{bc} = \sum \left(1 - \frac{w_a^2}{bc}\right) = \sum \frac{bc - w_a^2}{bc} = \sum \left(\frac{a}{b+c}\right)^2 \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{a}{b+c}\right)^2}{3} \overset{NEBITT}{\geq}$$

$$\overset{NESBITT}{\geq} \frac{1}{3}\cdot\left(\frac{3}{2}\right)^2 = \frac{3}{4} \Leftrightarrow 3 - \sum \frac{w_a^2}{bc} \geq \frac{3}{4} \Leftrightarrow -\sum \frac{w_a^2}{bc} \geq \frac{3}{4} - 3 = -\frac{9}{4} \Leftrightarrow \sum \frac{w_a^2}{bc} \leq \frac{9}{4}.$$

The last inequality is inequality to prove. The proof is complete.

PP. 19841. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum_{k=1}^{n} \frac{a_k^2}{(a_k+1)(a_k^3+1)} \leq \frac{n}{4}$

2). $\sum_{k=1}^{n} \frac{a_k^3}{(a_k^2+a_k+1)(a_k^4+1)} \leq \frac{n}{6}$

Mihály Bencze

We have

$$1) \sum_{k=1}^{n} \frac{a_k^2}{(a_k+1)(a_k^3+1)} \leq \sum_{k=1}^{n} \frac{a_k^2}{2\sqrt{a_k} \cdot 2\sqrt{a_k^3}} = \frac{1}{4}\sum_{k=1}^{n} 1 = \frac{n}{4};$$

$$2) \sum_{k=1}^{n} \frac{a_k^3}{(a_k^2+a_k+1)(a_k^4+1)} \leq \sum_{k=1}^{n} \frac{a_k^3}{3 \cdot \sqrt[3]{a_k^2 \cdot a_k \cdot 1} \cdot 2 \cdot \sqrt{a_k^4 \cdot 1}} = \frac{1}{6}\sum_{k=1}^{n} 1 = \frac{n}{6},$$

and the proof is complete.

PP. 19861. In all triangle ABC holds $\sum \frac{\sin^4 \frac{A}{2}}{b^2c^2\left(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}\right)} \geq \frac{1}{(4R+r)R^3}$.

Mihály Bencze

By *Bergström's* inequality and well-known formulas we have

$$\sum \frac{\sin^4 \frac{A}{2}}{b^2c^2(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2})} \overset{BERGSTROM}{\geq} \frac{\left(\sum \frac{\sin^2 \frac{A}{2}}{bc}\right)^2}{\sum\left(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}\right)} = \frac{\left(\sum \frac{\sin^2 \frac{A}{2}}{bc}\right)^2}{2\sum\left(\cos^2 \frac{A}{2}\right)} =$$

$$= \frac{\left(\sum \frac{\sin^2 \frac{A}{2}}{bc}\right)^2}{2 \cdot \frac{4R+r}{2R}} = \frac{R}{4R+r}\left(\sum \frac{\sin^2 \frac{A}{2}}{bc}\right)^2 = \frac{R}{4R+r} \cdot \frac{1}{R^4} = \frac{1}{(4R+r)R^3}, \text{ q.e.d}$$

PP. 19863. Prove that $\sum_{k=1}^{n} \frac{k^4}{k^2+k+1} \geq \frac{3n(n+1)^2}{4(n^2+3n+5)}$.

Mihály Bencze

Using *Bergström's* inequality and well-known formulas for sums we have

$$\sum_{k=1}^{n} \frac{k^4}{k^2+k+1} = \sum_{k=1}^{n} \frac{(k^2)^2}{k^2+k+1} \overset{BERGSTROM}{\geq} \frac{\left(\sum_{k=1}^{n} k^2\right)^2}{\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + n} = \frac{3n(n+1)^2}{4(n^2+4n+5)},$$

and we are done.

PP. 19895. In all triangle ABC holds

1). $\sum \frac{1}{r_a^2+2r_br_c} \geq \frac{3}{(4R+r)^2-2s^2}$

2). $\sum \frac{1}{(r_a^2+2r_br_c)(r_b^2+2r_cr_a)} \geq \frac{3}{\left((4R+r)^2-2s^2\right)^2}$

3). $\left(r_a^2+2r_br_c\right)\left(r_b^2+2r_cr_a\right) \leq \left((4R+r)^2-2s^2\right)^3$

Mihály Bencze

By Meens inequality, *Bergström*' s inequality and well-known formulas we have

$$1) \sum \frac{1}{r_a^2+2r_br_c} \geq \sum \frac{1}{r_a^2 + 2\cdot\frac{r_b^2+r_c^2}{2}} = \sum \frac{1}{r_a^2+r_b^2+r_c^2} =$$

$$= \frac{3}{r_a^2+r_b^2+r_c^2} = \frac{3}{(4R+r)^2-2s^2};$$

$$2) \sum \frac{1}{(r_a^2+2r_br_c)(r_b^2+2r_cr_a)} \geq \sum \frac{1}{\left(r_a^2+2\cdot\frac{r_b^2+r_c^2}{2}\right)\left(r_b^2+2\cdot\frac{r_c^2+r_a^2}{2}\right)} =$$

$$= \sum \frac{1}{(r_a^2+r_b^2+r_c^2)^2} = \frac{3}{(r_a^2+r_b^2+r_c^2)^2} = \frac{3}{\left((4R+r)^2-2s^2\right)^2},$$

and this completes the proof.

PP. 19896. In all triangle ABC holds $\sum \frac{a^2\left(\frac{1}{b}+\frac{1}{c}\right)^{*}}{r_b^2+r_c^2} \geq \frac{8(4R+r)^2}{s^2R^2\left((4R+r)^2-2s^2\right)}$.

Mihály Bencze

By *Bergström's* inequality and well-known formulas we have

$$\sum \frac{a^2\left(\frac{1}{b}+\frac{1}{c}\right)^4}{r_b^2+r_c^2} \geq \frac{\left(\sum a\left(\frac{1}{b}+\frac{1}{c}\right)^2\right)^2}{\sum (r_b^2+r_c^2)} = \frac{\left(\sum a\left(\frac{1}{b}+\frac{1}{c}\right)^2\right)^2}{2(r_a^2+r_b^2+r_c^2)} \geq$$

$$\geq \frac{1}{2\left((4R+r)^2-2s^2\right)}\left(\frac{4(4R+r)^2}{sR}\right)^2 = \frac{8(4R+r)^2}{s^2R^2\left((4R+r)^2-2s^2\right)},$$

and we are done.

PP. 19927. Let M be a point in the plane of triangle ABC. Denote A_1, B_1, C_1 the midpoints of sides BC, CA, AB, then prove that:

1). $\sum aMA_1^2 \geq sRr$

2). $\sum \frac{a^2MA_1^4}{b} \geq \frac{sR^2r^2}{2}$

3). $\sum \frac{a^2MA_1^4}{(s-b)^2} \geq \frac{s^2R^2r^2}{s^2-2r^2-8Rr}$

4). $\sum \frac{a^2MA_1^4}{r_br_c} \geq R^2r^2$

5). $\sum \frac{a^2MA_1^4}{\cos^2\frac{B}{2}} \geq \frac{2s^2R^3r^2}{4R+r}$

6). $\sum \frac{a^2MA_1^4}{\sin^2\frac{B}{2}} \geq \frac{2s^2R^3r^2}{2R-r}$

Mihály Bencze

By well-known formulas and *Bergström's* inequality we have

1) $$\sum aMA_1^2 = \sum \frac{a^2MA_1^2}{a} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1\right)^2}{\sum a} = \frac{\left(\sum aMA_1\right)^2}{2s} \geq \frac{2s^2Rr}{2s} = sRr;$$

2) $$\sum \frac{a^2MA_1^4}{b} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1^2\right)^2}{\sum a} \geq \frac{s^2R^2r^2}{2s} = \frac{sR^2r^2}{2};$$

3) $$\sum \frac{a^2MA_1^4}{(s-b)^2} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1^2\right)^2}{\sum (s-b)^2} \geq \frac{s^2R^2r^2}{s^2-2r^2-8Rr};$$

4) $$\sum \frac{a^2MA_1^4}{r_br_c} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1^2\right)^2}{\sum r_ar_b} \geq \frac{s^2R^2r^2}{s^2} = R^2r^2;$$

5) $$\sum \frac{a^2MA_1^4}{\cos^2\frac{B}{2}} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1^2\right)^2}{\sum \cos^2\frac{B}{2}} \geq \frac{s^2R^2r^2}{\frac{4R+r}{2R}} = \frac{2s^2R^3r^2}{4R+r};$$

6) $$\sum \frac{a^2 MA_1^4}{\sin^2 \frac{B}{2}} \overset{BERGSTROM}{\geq} \frac{\left(\sum aMA_1^2\right)^2}{\sum \sin^2 \frac{B}{2}} \geq \frac{s^2R^2r^2}{\frac{2R-r}{2R}} = \frac{2s^2R^3r^2}{2R-r}.$$

The proof is complete.

PP. 19931. In all triangle ABC holds $\sum \frac{a^2+r_a^2}{m_a^2} \geq \frac{4\left(2s^2+(4R+r)^2\right)}{3(s^2-r^2-4Rr)}$.

Mihály Bencze

By well-known formulas and *Bergström's i*nequality we have

$$\sum \frac{a^2+r_a^2}{m_a^2} \overset{BERGSTROM}{\geq} \frac{(a+b+c)^2+(r_a+r_b+r_c)^2}{\sum m_a^2} =$$

$$= \frac{4s^2+(4R+r)^2}{\frac{3}{4}(a^2+b^2+c^2)} = \frac{4\left(4s^2+(4R+r)^2\right)}{3\cdot 2(s^2-r^2-4Rr)} = \frac{2\left(4s^2+(4R+r)^2\right)}{3(s^2-r^2-4Rr)}, \text{ q.e.d.}$$

3. İNEGALITATEA DİNTRE MEDİA ARİTMETİCĂ ŞI MEDİA GEOMETRİCĂ

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ŞCOALA GİMNAZİALĂ PÎRSCOV, JUD. BUZĂU

İnegalitatea dintre media aritmetică şi media geometrică este:

$$\frac{a+b}{2} \geq \sqrt{ab}, \forall\, a,b \geq 0$$

Demonstraţie:

$$\frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow \frac{a+b}{2} - \sqrt{ab} \geq 0 \Leftrightarrow \frac{a+b-2\sqrt{ab}}{2} \geq 0 \Leftrightarrow$$

$\frac{(\sqrt{a})^2 - 2\sqrt{a}\cdot\sqrt{b} + (\sqrt{b})^2}{2} \geq 0 \Leftrightarrow \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$, ceea ce este evident.

Observaţie: Dacă $a = b$, atunci inegalitatea devine $\frac{a+a}{2} \geq \sqrt{a \cdot a} \Leftrightarrow \frac{2a}{2} \geq \sqrt{a^2} \Leftrightarrow a \geq |a|$, ceea ce este adevărat.

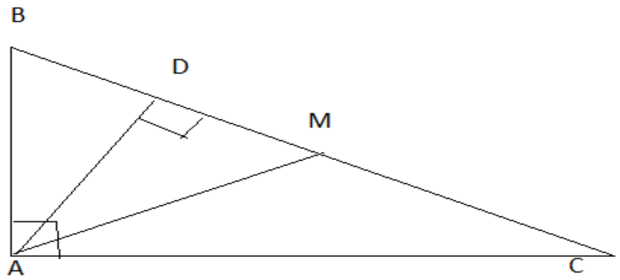
Aplicaţii:

Vom verifica inegalitatea şi pentru $a = x$ şi $b = \frac{1}{x}, \forall x > 0$

$$\frac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} \Leftrightarrow \frac{x+\frac{1}{x}}{2} \geq \sqrt{1} \Leftrightarrow x + \frac{1}{x} \geq 2 \Leftrightarrow x - 2 + \frac{1}{x} \geq 0 \Leftrightarrow \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0.$$

Vom demonstra geometric inegalitatea dintre media aritmetică şi cea geometrică:

Considerăm un **triunghi dreptunghic** cu $m(\hat{A}) = 90^0$, $AD \perp BC,\ D \in BC$, $CD = a,\ \ DB = b,$ şi mediana AM



$AM = \frac{a+b}{2}$ (1)

$AD = \sqrt{ab}$ (2)

Din relaţiile (1) şi (2) rezultă $AM \geq AD$ (egalitatea are loc pentru $M = D$, adică $a = b$.

Deci $m_a \geq m_g$.

Aplicaţii ale inegalităţii dintre media aritmetică şi media geometrică în determinarea maximului sau minimului unei expresii algebrice

1. Aflaţi valoarea minimă a expresiei:

$$E(x) = \frac{x^2+2x+26}{x+1}.$$

<u>*Soluţie*:</u>

$$E(x) = \frac{x^2 + 2 \cdot x + 1 + 25}{x+1}$$

$$E(x) = \frac{(x+1)^2 + 25}{x+1}$$

$$E(x) = (x+1) + \frac{25}{x+1}$$

$$\frac{x+1+\frac{25}{x+1}}{2} \geq \sqrt{(x+1) \cdot \frac{25}{x+1}}$$

$$\frac{E(x)}{2} \geq \sqrt{25}$$

$E(x) \geq 2 \cdot 5$

$E(x) \geq 10 \Rightarrow minE(x) = 10.$

2. Aflaţi valoarea maximă a expresiei:

$$E(x) = \sqrt{-9x^2 + 3x + 56}$$

Soluţie:

$E(x) = \sqrt{-9x^2 + 24x - 21x + 56}$

$E(x) = \sqrt{3x(-3x + 8) + 7(-3x + 8)}$

$E(x) = \sqrt{(3x + 7)(8 - 3x)}$

$\sqrt{(3x + 7)(8 - 3x)} \leq \dfrac{(3x + 7) + (8 - 3x)}{2}$

$\sqrt{(3x + 7)(8 - 3x)} \leq \dfrac{15}{2} \Rightarrow E(x) \leq 7{,}5 \Rightarrow \max E(x) = 7{,}5$

3. Demonstraţi că pentru orice $x, y, z \in R$ avem:

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Soluţie:

$\frac{x^2+y^2}{2} \geq \sqrt{x^2y^2} = |xy| \geq xy$ (1)

$\frac{y^2+z^2}{2} \geq \sqrt{y^2z^2} = |yz| \geq yz$ (2)

$\frac{x^2+z^2}{2} \geq \sqrt{x^2z^2} = |xz| \geq xz$ (3)

Adunând membru cu membru obținem:

$$\frac{x^2+y^2}{2}+\frac{y^2+z^2}{2}+\frac{x^2+z^2}{2}\geq xy+yz+xz$$

$$\frac{2(x^2+y^2+z^2)}{2}\geq xy+yz+xz\Rightarrow x^2+y^2+z^2\geq xy+yz+xz.$$

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