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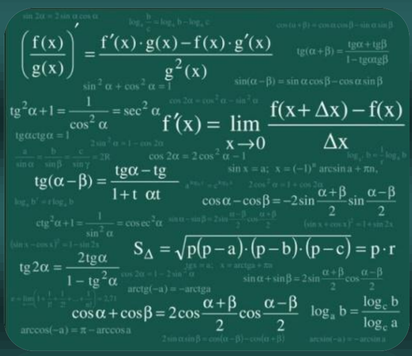
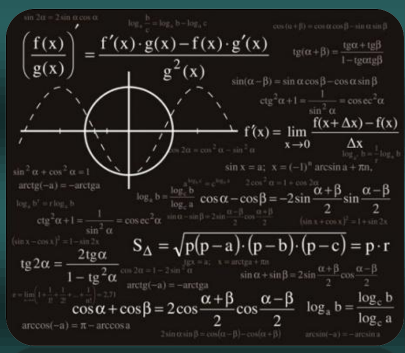

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REDACTORI PRINCIPALI ŞI SUSŢINĂTOR PERMANENŢI AI REVISTEI

NECULAI STANCIU, ROXANA MIHAELA STANCIU ŞI NELA CICEU

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1. Solutions and hints from some problems from Octogon Mathematical Magazine (VI)

by D.M. Bătineţu-Giurgiu,
National College ''Matei Basarab'', Bucharest, Romania
and
Neculai Stanciu,
''George Emil Palade'' School, Buzău, Romania

PP.20897. In all triangle holds $\sum a^3(b^2+c^2)\geq 144sR^2r^2$.

Solution. By AM-GM inequality we obtain:

$$\sum a^3(b^2+c^2)\geq 2\sum a^3bc=2abc\sum a^2=8Rrs\sum a^2=4Rr\left(\sum a\right)\left(\sum a^2\right)\geq$$

$$\geq 4Rr\cdot 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}=36Rr\cdot abc=144sR^2r^2.$$

PP.20906. Let $ABCDEFGIJK$ be a regular 11-gon. Prove that:

1) $AE^2-AD^2=AB\cdot AE$;

2) $\dfrac{AD}{AB}-\dfrac{AE}{AC}=1$.

Solution. Let O be the circumcenter of the polygon. We assume that the radius of circumscribed circle is equal with 1. Because

$$\angle AOB=\frac{2\pi}{11},\angle AOD=\frac{6\pi}{11},\angle AOE=\frac{8\pi}{11},$$

we have:

$$AB=2\sin\frac{\pi}{11},\ AE=2\sin\frac{4\pi}{11},\ AD=2\sin\frac{3\pi}{11},\ AC=2\sin\frac{2\pi}{11}.$$

1) We obtain: $AE^2-AD^2=AB\cdot AE\Leftrightarrow \sin^2\frac{4\pi}{11}-\sin^2\frac{3\pi}{11}=\sin\frac{\pi}{11}\cdot\sin\frac{4\pi}{11}\Leftrightarrow$

$$\Leftrightarrow 1-\cos\frac{8\pi}{11}-1+\cos\frac{6\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{4\pi}{11}\Leftrightarrow\cos\frac{6\pi}{11}-\cos\frac{8\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{4\pi}{11}$$

$\Leftrightarrow 2\sin\frac{7\pi}{11}\sin\frac{\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{4\pi}{11}$, which is true because $\sin\frac{7\pi}{11}=\sin\frac{4\pi}{11}$.

2) $\dfrac{AD}{AB}-\dfrac{AE}{AC}=1\Leftrightarrow 2\sin\frac{3\pi}{11}\sin\frac{2\pi}{11}-2\sin\frac{\pi}{11}\sin\frac{4\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{2\pi}{11}$

$$\Leftrightarrow\cos\frac{\pi}{11}-\cos\frac{5\pi}{11}-\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{2\pi}{11}$$

$\Leftrightarrow \cos\frac{\pi}{11}-\cos\frac{3\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{2\pi}{11} \Leftrightarrow 2\sin\frac{2\pi}{11}\sin\frac{\pi}{11}=2\sin\frac{\pi}{11}\sin\frac{2\pi}{11}$, true, and we are done.

PP.20907. If $a,b,c>0$, then $\sum\frac{a^3}{\sqrt{2a(a+b)^3}+\sqrt{2b^2(a^2+b^2)}}\geq\frac{1}{6}\sum a$.

Solution. By AM-GM inequality we have:

$$\sqrt{2a(a+b)^3}+\sqrt{2b^2(a^2+b^2)}=(a+b)\sqrt{2a(a+b)}+\sqrt{2b^2(a^2+b^2)}\leq$$

$$\leq\frac{(a+b)(2a+a+b)}{2}+\frac{2b^2+a^2+b^2}{2}=2a^2+2ab+2b^2,$$

so by this and by Harald Bergström' s inequality yields that:

$$\sum\frac{a^3}{\sqrt{2a(a+b)^3}+\sqrt{2b^2(a^2+b^2)}}\geq\frac{1}{2}\sum\frac{a^3}{a^2+ab+b^2}=\frac{1}{2}\sum\frac{a^4}{a^3+a^2b+ab^2}\geq$$

$$\geq\frac{1}{2}\cdot\frac{\left(\sum a^2\right)^2}{\sum a^3+\sum a^2b+\sum ab^2}.$$

Therefore, it remains to prove that:

$$3\left(\sum a^2\right)^2\geq\sum a\sum a^3+\sum a\sum a^2b+\sum a\sum ab^2$$

$$\Leftrightarrow 3\sum a^4+6\sum a^2b^2\geq\sum a^4+\sum a^3b+\sum ab^3+\sum a^3b+\sum a^2bc+\sum a^2b^2+$$

$$+\sum ab^3+\sum a^2b^2+\sum a^2bc$$

$$\Leftrightarrow\sum a^4+2\sum a^2b^2\geq\sum a^3b+\sum ab^3+\sum a^2bc \qquad (1)$$

The inequality (1) yields from adding the following inequalities (which was obtained by AM-GM inequality):

$$a^4+a^2b^2\geq 2a^3b;\ b^4+b^2c^2\geq 2b^3c;\ c^4+c^2a^2\geq 2ac^3;$$

$$a^4+a^2c^2\geq 2a^3c;\ b^4+b^2a^2\geq 2b^3a;\ c^4+c^2b^2\geq 2bc^3;$$

$$a^2b^2+c^2a^2\geq 2a^2bc;\ a^2b^2+c^2b^2\geq 2ab^2c;\ b^2c^2+c^2a^2\geq 2abc^2.$$

The proof is complete.

PP.20910. If $x_k>0$ $(k=1,2,...,n)$, then

$$\sum_{cyclic}\frac{(x_1+x_2)(x_2+x_3)(x_3+x_4)(x_4+x_1)}{x_1x_2x_3+x_2x_3x_1+x_3x_4x_1+x_4x_1x_2}\geq 4\sum_{k=1}^{n}x_k.$$

Solution. We have the inequality $\frac{(a+b)(b+c)(c+d)(d+a)}{abc+bcd+cda+dab}\geq a+b+c+d$ which after some algebra becomes $a^2c^2+b^2d^2\geq 2abcd$, true.

Therefore:

$$\sum_{cyclic}\frac{(x_1+x_2)(x_2+x_3)(x_3+x_4)(x_4+x_1)}{x_1x_2x_3+x_2x_3x_1+x_3x_4x_1+x_4x_1x_2}\geq\sum_{cyclic}(x_1+x_2+x_3+x_4)=4\sum_{k=1}^{n}x_k .$$

PP.20912. Let $ABCDEFGHIJKLM$ be a regular 13-gon. Prove that

$$\frac{AC-AB}{AD-AC}=\frac{AF}{AE}.$$

Solution. Let O be the circumcenter with the radius $\frac{1}{2}$.

Since $\angle AOB=\frac{2\pi}{13}$, denoting $\alpha=\frac{\pi}{13}$, we have $AB=\sin\alpha$, $AC=\sin 2\alpha$, $AD=\sin 3\alpha$, $AE=\sin 4\alpha$, $AF=\sin 5\alpha$.

Yields

$$\frac{AC-AB}{AD-AC}=\frac{AF}{AE}\Leftrightarrow\frac{\sin 2\alpha-\sin\alpha}{\sin 3\alpha-\sin\alpha}=\frac{\sin 5\alpha}{\sin 4\alpha}\Leftrightarrow\frac{2\sin\frac{\alpha}{2}\cos\frac{3\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{5\alpha}{2}}=\frac{\sin 5\alpha}{\sin 4\alpha}$$

$$\Leftrightarrow 2\sin 4\alpha\sin\frac{3\alpha}{2}=2\sin 5\alpha\cos\frac{5\alpha}{2}$$

$\Leftrightarrow \sin\frac{11\alpha}{2}+\sin\frac{5\alpha}{2}=\sin\frac{15\alpha}{2}+\sin\frac{5\alpha}{2}\Leftrightarrow\sin\frac{11\pi}{26}=\sin\frac{15\pi}{26}$, true because $\frac{11\pi}{26}+\frac{15\pi}{26}=\pi$, and we are done.

PP.20917. If $a,b,c,d>0$, then

$$\left(\frac{a}{b}+1\right)\left(\frac{b}{c}+1\right)\left(\frac{c}{d}+1\right)\left(\frac{d}{a}+1\right)\geq(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16.$$

Solution. After some algebra the first inequality becomes $\frac{ac}{bd}+\frac{bd}{ac}\geq 2$, true by AM-GM inequality. The second inequality is the AM-HM inequality (or Cauchy-Buniakovski-Schwarz inequality). The proof is complete.

PP.20920. If $a_k>0$ $(k=1,2,...,n)$, then $\prod_{k+1}^{n}(a_k^n+n-1)\geq\left(\sum_{k=1}^{n}a_k\right)^n$.

Solution. By Hölder's inequality we obtain:

$$\prod_{k+1}^{n}(a_k^n+n-1)=(a_1^n+1+...+1)(1+a_2^n+1+...+1)\cdot...\cdot(1+...+1+a_n^n)\geq$$

$$\geq\left(\sqrt[n]{a_1^n\cdot 1\cdot\ldots\cdot 1}+\sqrt[n]{1\cdot a_2^n\cdot\ldots\cdot 1}+\ldots+\sqrt[n]{1\cdot\ldots\cdot 1\cdot a_n^n}\right)^n=\left(\sum_{k=1}^n a_k\right)^n$$
, and we are done.

PP.20928. If $x_k>0\ \ (k=1,2,...,n)$ and $\sum_{k=1}^n x_k=1$, then $\sum_{k=1}^n\frac{1}{1+x_k^2}\leq\frac{n^3}{n^2+1}$.

Solution. For $n=1$, we have equality. For $n=2$ we have:

$$\frac{1}{1+x^2}+\frac{1}{1+(1-x)^2}\leq\frac{8}{5}\Leftrightarrow 8x^4-16x^3+14x^2-6x+1\geq 0\Leftrightarrow$$

$\Leftrightarrow (2x-1)^2[x^2+(x-1)^2]\geq 0$, true.

We prove that: if $x\leq 1$ and $n\geq 3$, then:

$$\frac{1}{1+x^2}\leq\frac{n^2}{n^2+1}-\frac{2n^3}{(n^2+1)^2}\left(x-\frac{1}{n}\right)\qquad (*)$$

$\Leftrightarrow (nx-1)^2(2nx-n^2+1)\leq 0$, true because if $x\leq 1$ and $n\geq 3$, then:

$$x\leq 1<\frac{n^2-1}{2n}\Rightarrow 2nx-n^2+1<0.$$

Writing the inequality (*) for $x_1,x_2,...,x_n$ and adding where we taking account that $\sum_{k=1}^n x_k=1$ we obtain the desired result.

PP.20931. If $a_k>0\ \ (k=1,2,...,n)$, then $\sum_{cyclic}\frac{a_1^3}{a_2^2-a_2a_3+a_3^2}\geq\sum_{k=1}^n a_k$.

Solution. We will prove that:

$$\sum_{cyclic}\left(\frac{a_1^3}{a_2^2-a_2a_3+a_3^2}+\frac{a_2^3}{a_3^2-a_3a_1+a_1^2}+\frac{a_3^3}{a_1^2-a_1a_2+a_2^2}\right)\geq 3\sum_{k=1}^n a_k.$$

In this aim, we show that for $x,y,z>0$, holds:

$$\frac{x^3}{y^2-yz+z^2}+\frac{y^3}{z^2-zx+x^2}+\frac{z^3}{x^2-xy+y^2}\geq x+y+z\qquad (1)$$

(i.e. problem 24477 from Gazeta matematică 3/2001, proposed by Ion Nedelcu, with restriction x,y,z to be the lengths of the sides of a triangle). We prove that (1) occurs for all $x,y,z>0$. Indeed,

$$\sum\frac{x^3}{y^2-yz+z^2}-\sum x=\sum\left(\frac{x^3(y+z)}{y^3+z^3}-x\right)=\sum\frac{x^3y+x^3z-xy^3-xz^3}{y^3+z^3}=$$

$$=\sum\frac{xy(x^2-y^2)}{y^3+z^3}+\sum\frac{xz(x^2-z^2)}{y^3+z^3}=\sum\frac{xy(x^2-y^2)}{y^3+z^3}+\sum\frac{xy(y^2-x^2)}{z^3+x^3}=$$

$$=\sum xy(x^2-y^2)\left(\frac{1}{y^3+z^3}-\frac{1}{z^3+x^3}\right)=\sum\frac{xy(x-y)(x^3-y^3)}{(y^3+z^3)(z^3+x^3)}\geq 0$$
, and (1) is proved.

Using the inequality (1) yields that:
$\sum_{cyclic}\left(\frac{a_1^3}{a_2^2-a_2a_3+a_3^2}+\frac{a_2^3}{a_3^2-a_3a_1+a_1^2}+\frac{a_3^3}{a_1^2-a_1a_2+a_2^2}\right)\geq\sum_{cyclic}(a_1+a_2+a_3)=3\sum_{k=1}^{n}a_k$, and we are done.

PP.20936. If $a_k>0\ (k=1,2,...,n)$, then $\sum_{cyclic}\frac{a_1}{a_2+2a_3}\geq\frac{n}{3}$.

Solution. We will prove that:

$$\sum_{cyclic}\left(\frac{a_1}{a_2+2a_3}+\frac{a_2}{a_3+2a_1}+\frac{a_3}{a_1+2a_2}\right)\geq n.$$

We have the inequality;

$$\frac{x}{y+2z}+\frac{y}{z+2x}+\frac{z}{x+2y}\geq 1 \qquad (1)$$

Indeed, by Bergsytröm's inequality and well-known $\left(\sum x\right)^2\geq 3\sum xy$, we obtain:

$$\frac{x}{y+2z}+\frac{y}{z+2x}+\frac{z}{x+2y}=\frac{x^2}{xy+2xz}+\frac{y^2}{yz+2xy}+\frac{z^2}{xz+2yz}\geq\frac{\left(\sum x\right)^2}{3\sum xy}\geq 1.$$

Yields that: $\sum_{cyclic}\left(\frac{a_1}{a_2+2a_3}+\frac{a_2}{a_3+2a_1}+\frac{a_3}{a_1+2a_2}\right)\geq\sum_{cyclic}1=n$, and we are done.

PP.20938. If $x_i>0\ (i=1,2,...,n)$ and $k\in N^*-\{1\}$, then:

$$\sum_{i=1}^{n}x_i^k+k\sum_{i=1}^{n}\sqrt[k]{x_i}\geq(k+1)\sum_{i=1}^{n}x_i.$$

Solution. Applying AM-GM inequality, we obtain:

$$a^k+k\cdot\sqrt[k]{a}=a^k+\underbrace{\sqrt[k]{a}+\sqrt[k]{a}+...+\sqrt[k]{a}}_{k}\geq(k+1)\cdot\sqrt[k+1]{a^k\left(\sqrt[k]{a}\right)^k}=(k+1)\cdot\sqrt[k+1]{a^{k+1}}=(k+1)a.$$

Using the inequality $a^k+k\cdot\sqrt[k]{a}\geq(k+1)a$ for numbers $x_1,x_2,...,x_n$ and adding yields the inequality from the statement. The proof is complete.
Remark. The inequality $a^k+k\cdot\sqrt[k]{a}\geq(k+1)a$ can be proved without AM-GM inequality, so can be used to prove by mathematical induction AM-GM inequality.

PP.20944. If $a,b,c>0$, then $\sum\frac{a}{b}\geq 2+\frac{\sum a^2}{\sum ab}$.

Solution 1. By Cauchy-Buniakovski-Schwarz inequality we have:
$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(ab+bc+ca)\geq(a+b+c)^2$, and then:

$$\sum \frac{a}{b} \geq \frac{\sum a^2 + 2\sum ab}{\sum ab} = 2 + \frac{\sum a^2}{\sum ab}.$$

Solution 2. By Bergström's inequality we obtain:

$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a^2}{ab} + \frac{b^2}{bc} + \frac{c^2}{ac} \geq \frac{(a+b+c)^2}{ab+bc+ca} = 2 + \frac{\sum a^2}{\sum ab}$, and we are done.

PP.20945. If $x_k > 0 \ (k = 1,2,...,n)$, then:

$$\sum_{cyclic} \sqrt{1 + \frac{6}{x_1 + x_2 + x_3} + \frac{3}{x_1 x_2 + x_2 x_3 + x_3 x_1}} \geq n + \frac{n^2}{\sum_{k=1}^{n} x_k}.$$

Solution. Since $x_1 x_2 + x_2 x_3 + x_3 x_1 \leq \frac{(x_1 + x_2 + x_3)^2}{3}$, and using AM-HM inequality (or Bergström's inequality) we obtain that:

$$\sum_{cyclic} \sqrt{1 + \frac{6}{x_1 + x_2 + x_3} + \frac{3}{x_1 x_2 + x_2 x_3 + x_3 x_1}} \geq \sum_{cyclic} \sqrt{1 + \frac{6}{x_1 + x_2 + x_3} + \frac{9}{(x_1 + x_2 + x_3)^2}} =$$

$$= \sum_{cyclic} \left(1 + \frac{3}{x_1 + x_2 + x_3}\right) = n + 3 \cdot \sum_{cyclic} \frac{1}{x_1 + x_2 + x_3} \geq n + 3 \cdot \frac{n^2}{\sum_{cyclic} (x_1 + x_2 + x_3)} =$$

$= n + 3 \cdot \frac{n^2}{3 \cdot \sum_{k=1}^{n} x_k} = n + \frac{n^2}{\sum_{k=1}^{n} x_k}$, and the proof is complete.

PP.20946. If $a,b,c > 0$, then $6(\sum a)(\sum ab) \leq 18abc + (\sum a)(\sum a^2) + (\sum a)^3$.

Solution. The inequality from the statement is written successively:

$6\sum a^2 b + 6\sum ab^2 + 18abc \leq 18abc + \sum a^3 + \sum a^2 b + \sum ab^2 + \sum a^3 + 3\sum a^2 b +$
$+ 3\sum ab^2 + 6abc \Leftrightarrow 2\sum a^3 + 6abc \geq 2\sum a^2 b + 2\sum ab^2$
$\Leftrightarrow \sum a^3 + 3abc \geq \sum a^2 b + \sum ab^2$, which is Schur's inequality. The proof is complete.

PP.20949. If $a_k > 0 \ (k = 1,2,...,n)$ and $s = \sum_{k=1}^{n} a_k$, then:

$$n(n-1)\sum_{k=1}^{n} a_k^3 \geq \left(\sum_{k=1}^{n} a_k \sqrt{s - a_k}\right)^2.$$

Solution. We use Chebyshev' s inequality, Cauchy-Buniakovski-Schwarz inequality, and the fact that $\sum_{k=1}^{n}(s-a_k)=(n-1)\sum_{k=1}^{n}a_k$. Therefore, we obtain:

$$n(n-1)\sum_{k=1}^{n}a_k^3=(n-1)\cdot n\sum_{k=1}^{n}a_k^2\cdot a_k\geq(n-1)\sum_{k=1}^{n}a_k^2\cdot\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}a_k^2\cdot\sum_{k=1}^{n}(s-a_k)\geq$$

$$\geq\left(\sum_{k=1}^{n}a_k\sqrt{s-a_k}\right)^2$$, and we are done.

PP.20951. If $a_k>0\ (k=1,2,...,n)$ and $s=\sum_{k=1}^{n}a_k$, then:

$$\sum_{k=1}^{n}\frac{a_k}{(s-a_k)^2}\geq\frac{n^2}{(n-1)^2\sum_{k=1}^{n}a_k}.$$

Solution. First we prove that for $x\leq s$, we have the inequality:

$$\frac{x}{(s-x)^2}\geq\frac{n^2(n+1)}{s^2(n-1)^3}\cdot x-\frac{2n}{s(n-1)^3}\qquad(1)$$

The inequality (1) is written successively:

$$s^2(n-1)^3x\geq s^2n^2(n+1)x-2sn^2(n+1)x^2+n^2(n+1)x^3-2ns^3+2ns^2x+2nsx^2$$

$$\Leftrightarrow(n^3+n^2)x^3-2sn(n^2+n+1)x^2+s^2(4n^2-n+1)x-2ns^3\leq0$$

$$\Leftrightarrow\left(x-\frac{s}{n}\right)^2\left((n+1)x-2sn\right)\leq0$$, true because $x\leq s$ (thus $(n+1)s<2sn$).

We write the inequality (1) for $a_1,a_2,...,a_n$ an summing yields that:

$$\sum_{k=1}^{n}\frac{a_k}{(s-a_k)^2}\geq\frac{n^2(n+1)}{s^2(n-1)^3}\sum_{k=1}^{n}a_k-\frac{2n^2}{s(n-1)^3}=$$

$$=\frac{n^2(n+1)-2n^2}{s(n-1)^3}=\frac{n^2}{(n-1)^2\sum_{k=1}^{n}a_k}$$, and the proof is complete.

PP.20952. If $x_k>0\ (k=1,2,...,n)$, then:

$$\sum_{cyclic}\sqrt{x_1^4+x_1^2x_2^2+x_2^4}\geq\sqrt{3}\sum_{k=1}^{n}x_k^2\geq\sum_{cyclic}x_1\sqrt{2x_1^2+x_2x_3}.$$

Solution. Using the inequality $a^2+ab+b^2\geq\frac{3(a+b)^2}{4}$, we obtain:

$$\sum_{cyclic}\sqrt{x_1^4+x_1^2x_2^2+x_2^4}\geq\sum_{cyclic}\frac{\sqrt{3}(x_1^2+x_2^2)}{2}=\frac{\sqrt{3}}{2}\sum_{cyclic}(x_1^2+x_2^2)=\sqrt{3}\sum_{k=1}^{n}x_k^2.$$

For the right inequality we apply the inequality of means:

$$x_1\sqrt{2x_1^2+x_2x_3}=\frac{1}{\sqrt{3}}\cdot\sqrt{(3x_1^2)(2x_1^2+x_2x_3)}\le\frac{1}{\sqrt{3}}\cdot\frac{5x_1^2+x_2x_3}{2}\le$$

$$\le\frac{1}{\sqrt{3}}\cdot\frac{5x_1^2+\frac{x_2^2+x_3^2}{2}}{2}=\frac{1}{4\sqrt{3}}(10x_1^2+x_2^2+x_3^2)$$. Yields that:

$$\sum_{cyclic}x_1\sqrt{2x_1^2+x_2x_3}\le\frac{1}{4\sqrt{3}}\sum_{cyclic}(10x_1^2+x_2^2+x_3^2)=\frac{1}{4\sqrt{3}}\cdot 12\sum_{k=1}^{n}x_k^2=\sqrt{3}\sum_{k=1}^{n}x_k^2,$$

and the proof is complete.

PP.20957. In all nonisosceles triangle ABC holds:

$$\sum\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{r(3s^2-r^2-4Rr)}{4R^3}$$ (correction).

Solution. We use the formulas:

$$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2};\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s^2 4}{abc};$$

$$\sum\frac{a^4}{(a-b)(a-c)}=a^2+b^2+c^2+ab+bc+ca$$. We obtain that:

$$\sum\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\sum\frac{2\sin^4 A\sin\frac{A}{2}\cos\frac{A}{2}}{2\cdot\frac{a-b}{c}\cdot\frac{a-c}{b}\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=\frac{abc}{2s^2r}\sum\frac{bc\sin^5 A}{(a-b)(a-c)}=\frac{abc}{2s^2r}\sum\frac{bc\cdot a^5}{32R^5(a-b)(a-c)}=$$

$$=\frac{a^2b^2c^2}{64s^2rR^5}\sum\frac{a^4}{(a-b)(a-c)}=\frac{16R^2r^2s^2}{64s^2rR^5}\left(\sum a^2+\sum ab\right)=$$

$$=\frac{r(2s^2-2r^2-8Rr+s^2+r^2+4Rr)}{4R^3}=\frac{r(3s^2-r^2-4Rr)}{4R^3}$$, and we are done.

PP.20958. In all nonisosceles triangle ABC holds:

$$\sum\frac{\sin^3 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{sr}{R^2}.$$

Solution. We proceed like in PP.20957, and in addition we use the fact that:

$$\sum\frac{a^3}{(a-b)(a-c)}=a+b+c$$. Therefore, we have that:

$$\sum\frac{\sin^3 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\sum\frac{2\sin^3 A\sin\frac{A}{2}\cos\frac{A}{2}}{2\cdot\frac{a-b}{c}\cdot\frac{a-c}{b}\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$=\frac{abc}{2s^2r}\sum\frac{bc\sin^4 A}{(a-b)(a-c)}=\frac{abc}{2s^2r}\cdot\frac{abc}{16R^4}\sum\frac{a^3}{(a-b)(a-c)}=\frac{16R^2r^2s^2}{2s^2r\cdot 16R^4}\cdot 2s=\frac{sr}{R^2}$, and we are done.

PP.20960. In all nonisosceles triangle ABC holds:

$$\sum\frac{\sin\frac{A}{2}}{\sin^2 A\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{R(s^2+r^2+4Rr)}{s^2r}.$$

Solution. Using well-known formuls in triangle we obtain:

$$\sin\frac{A-B}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}\cdot\frac{s(s-b)}{ac}}-\sqrt{\frac{(s-a)(s-c)}{ac}\cdot\frac{s(s-a)}{bc}}=$$

$$=\frac{s-b-s+a}{c}\cos\frac{C}{2}=\frac{a-b}{c}\cos\frac{C}{2}.$$

Yields that:

$$\frac{\sin\frac{A}{2}}{\sin^2 A\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{\sin\frac{A}{2}}{\sin A\cdot 2\sin\frac{A}{2}\cos\frac{B}{2}\cdot\frac{a-b}{c}\cos\frac{C}{2}\cdot\frac{a-c}{2}\cos\frac{B}{2}}=$$

$$=\frac{bc}{2(a-b)(a-c)\cdot\frac{a}{2R}\cdot\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

Because $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sqrt{\frac{s(s-a)s(s-b)s(s-c)}{a^2b^2c^2}}=\frac{s^2r}{abc}$, we obtain:

$$\sum\frac{\sin\frac{A}{2}}{\sin^2 A\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{Rabc}{s^2r}\sum\frac{bc}{a(a-b)(a-c)} \qquad (1)$$

After some algebra we also obtain that:

$b^2c^2(c-b)+c^2a^2(a-c)+a^2b^2(b-a)=(ab+bc+ca)(a-b)(b-c)(c-a)$, and then by (1) taking account by $ab+bc+ca=s^2+r^2+4Rr$, yields the relation from the statement, and we are done.

PP.20982. If $a,b,c,\lambda>0$ and $a+b+c=1$, then:

$$\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}} \geq \frac{1}{\sqrt{\lambda\sum ab+3(1-\lambda)abc}}.$$

Solution. By Hölder's inequality we have:

$$\left(\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}}\right)\left(\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}}\right)\left(\sum a[\lambda(b^2+c^2)+bc]\right) \geq \left(\sum a\right)^3.$$

Therefore, $\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}} \geq \frac{1}{\sqrt{\sum a[\lambda(b^2+c^2)+bc]}}$, and it suffices to prove that:

$\lambda\sum ab+3(1-\lambda)abc \geq \lambda\sum a(b^2+c^2)+3abc$, but

$\lambda\sum a\sum ab-3\lambda abc \geq \lambda\sum a(b^2+c^2) \Leftrightarrow \sum a\sum ab-3abc \geq \sum a(b^2+c^2)$, true because in fact $\sum a\sum ab-3abc=\sum a(b^2+c^2)$.

PP.20984. If $a_k>0\ \ (k=1,2,...,n)$, then $\sum_{cyclic} \frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}} \geq 1$.

Solution. We will prove that:

$$\sum_{cyclic}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right) \geq 3.$$

Indeed, we have the inequality:

$$\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}} \geq \frac{2a_3}{n(a_1+a_2)} \Leftrightarrow (n^2-4)(a_1-a_2)^2 \geq 0.$$

Then, by Nesbitt's inequality yields that:

$$\sum_{cyclic}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right) \geq$$

$$\geq \sum_{cyclic}\frac{2}{n}\left(\frac{a_1}{a_2+a_3}+\frac{a_2}{a_1+a_3}+\frac{a_3}{a_1+a_2}\right) \geq \frac{2}{n}\sum_{cyclic}\frac{3}{2}=\frac{2}{n}\cdot\frac{3n}{2}=3,$$ and we are done.

PP.20992. If $a_k>0\ \ (k=1,2,...,n)$ and $\sum_{k=1}^{n} a_k^2=n$, then $\sum_{k=1}^{n}\frac{1}{n+1-a_k} \leq 1$.

Solution. For $n=1$ we have $a_1=1$ and $\frac{1}{n+1-a_1}=1$, so we have equality.

For $n=2$ we have $x^2+y^2=2$, so $xy \leq \sqrt{2}$ and we must to show that:

$$\frac{1}{3-x}+\frac{1}{3-y}\le 1 \Leftrightarrow 9-3x-3y+xy\ge 6-x-y \Leftrightarrow 3-2x\ge y(2-x) \quad (1)$$

Because $xy\le\sqrt{2}$ we can squaring and (1) is equivalent with:

$$(3-2x)^2\ge y^2(2-x)^2 \Leftrightarrow (3-2x)^2\ge(2-x^2)(2-x)^2 \Leftrightarrow$$

$\Leftrightarrow x^4-4x^3+6x^2-4x+1\ge 0 \Leftrightarrow (x-1)^4\ge 0$, evidently true.

Let $n\ge 3$. We prove that for any x with $x\le\sqrt{n}$ we have the following inequality:

$$\frac{1}{n+1-x}\le\frac{x^2}{2n^2}+\frac{2n-1}{2n^2} \qquad (2)$$

$$\Leftrightarrow 2n^2\le x^2(n+1)-x^3+(2n-1)(n+1)-(2n-1)x$$

$$\Leftrightarrow x^3-(n+1)x^2+(2n-1)x-(n-1)\le 0$$

$$\Leftrightarrow (x-1)^2\left(x-(n+1)\right)\le 0.$$

The last inequality is true because for $n\ge 3$ and $x\le\sqrt{n}$ we have:

$\sqrt{n}\le n-1 \Leftrightarrow n^2-3n+1\ge 0 \Leftrightarrow n(n-3)+1\ge 0$.

Writing the inequality (2) for $a_1,a_2,...,a_n$ and summing yields that:

$\sum_{k=1}^{n}\frac{1}{n+1-a_k}\le\frac{1}{2n^2}\sum_{k=1}^{n}a_k^2+n\cdot\frac{2n-1}{2n^2}=\frac{1}{2n}+\frac{2n-1}{2n}=1$. The proof is complete.

PP.20994. If $a_k>0$ $(k=1,2,...,n)$ and $\sum_{k=1}^{n}\frac{1}{n+1-a_k}=1$, then $\sum_{k=1}^{n}\frac{1}{n+1-a_k^2}\le 1$.

Solution. For $x>0$ we have the following inequality:

$$\frac{x}{n-1+x^2}\le\frac{2-n}{n-1+x}+\frac{n-1}{n} \qquad (1)$$

$$\Leftrightarrow n(n-1)x+nx^2\le -n(n-2)(n-1)-n(n-2)x^2+(n-1)^3+(n-1)^2x+$$
$$+(n-1)^2x^2+(n-1)x^3$$

$\Leftrightarrow (n-1)x^3-(n-1)x^2-(n-1)x+(n-1)\ge 0 \Leftrightarrow (n-1)(x-1)^2(x+1)\ge 0$, true.

Writing the inequality (1) for $a_1,a_2,...,a_n$ and adding we obtain that:

$\sum_{k=1}^{n}\frac{1}{n+1-a_k^2}\le(2-n)\sum_{k=1}^{n}\frac{1}{n-1+a_k}+n\cdot\frac{n-1}{n}=2-n+n-1=1$, and we are done.

PP.20995. If $a_k>0$ $(k=1,2,...,n)$ and $\sum_{k=1}^{n}a_k=n$, then $\sum_{k=1}^{n}\frac{1}{\lambda+a_k^2}\le\frac{n}{1+\lambda}$.

Solution. The inequality is not true without an additional condition for λ.

Indeed, for $n=2$, $\lambda=\frac{1}{2}$, $a_1=\frac{1}{2}$, $a_2=\frac{3}{2}$ it should that

$$\frac{1}{\frac{1}{2}+\frac{1}{4}}+\frac{1}{\frac{1}{2}+\frac{9}{4}}\le\frac{2}{1+\frac{1}{2}}\Leftrightarrow\frac{4}{3}+\frac{4}{11}\le\frac{4}{3}, \text{ false.}$$

We give a solution for $\lambda\ge 2n+1$.

We have the inequality

$$(*)\quad \frac{1}{\lambda+x^2}\le\frac{\lambda+3}{(\lambda+1)^2}-\frac{2x}{(\lambda+1)^2},$$

which from some calculations is equivalent with

$(x-1)^2(2x-(\lambda-1))\le 0$, true for $x\le n$ (because $2n\le\lambda-1$).

Writing the inequality (*) for $a_1,a_2,...,a_n$, by summation yields that

$$\sum_{k=1}^{n}\frac{1}{\lambda+a_k^2}\le\frac{n(\lambda+3)}{(1+\lambda)^2}-\frac{2}{(1+\lambda)^2}\sum_{k=1}^{n}a_k=\frac{n\lambda+3n-2n}{(1+\lambda)^2}=\frac{n}{1+\lambda}, \text{ and we are done.}$$

PP.20998. If $a,b,c>0$, then $\sum\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\ge 1$.

Solution. We have:

$$\sum a(a^2+3ab+3b^2+2bc)=\sum a^3+3\sum a^2b+3\sum ab^2+6abc=\left(\sum a\right)^3.$$

Applying Hölder's inequality we obtain:

$$\left(\sum\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\right)\left(\sum\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\right)\left(\sum a(a^2+3ab+3b^2+2bc)\right)\ge$$

$$\ge\left(\sum\sqrt[3]{\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\cdot\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\cdot a(a^2+3ab+3b^2+2bc)}\right)^3$$

$$\Leftrightarrow\left(\sum\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\right)^2\left(\sum a\right)^3\ge\left(\sum a\right)^3, \text{ i.e.}$$

$$\sum\frac{a}{\sqrt{a^2+3ab+3b^2+2bc}}\ge 1, \text{ and the proof is complete.}$$

PP.21000. If $a,b,c>0$, then $\left(\sum\frac{a}{\sqrt[n]{a+2b}}\right)^n\ge\left(\sum a\right)^{n-1}$, when $n\in N$, $n\ge 2$.

Solution. Applying Hölder's inequality we have:

$$\underbrace{\left(\sum\frac{a}{\sqrt[n]{a+2b}}\right)\left(\sum\frac{a}{\sqrt[n]{a+2b}}\right)...\left(\sum\frac{a}{\sqrt[n]{a+2b}}\right)}_{n}\left(\sum a(a+2b)\right)\ge\left(\sum a\right)^{n+1}, \text{ but}$$

$$\sum a(a+2b)=\left(\sum a\right)^2, \text{ so } \left(\sum\frac{a}{\sqrt[n]{a+2b}}\right)^n\ge\left(\sum a\right)^{n-1}, \text{ and we are done.}$$

PP.21001. If $a,b,c>0$, then $\left(\sum \frac{a}{\sqrt[n]{a^2+3ab+3b^2+2bc}}\right)^n \geq \left(\sum a\right)^{n-2}$, when $n\in N,\ n\geq 2$.

Solution. We have $\sum a(a^2+3ab+3b^2+2bc)=\left(\sum a\right)^3$, and by Hölder's inequality we obtain:

$$\underbrace{\left(\sum \frac{a}{\sqrt[n]{a^2+3ab+3b^2+2bc}}\right)\left(\sum \frac{a}{\sqrt[n]{a^2+3ab+3b^2+2bc}}\right)\ldots\left(\sum \frac{a}{\sqrt[n]{a^2+3ab+3b^2+2bc}}\right)}_{n}\cdot$$

$$\cdot\left(\sum a(a^2+3ab+3b^2+2bc)\right)\geq \left(\sum a\right)^{n+1} \Leftrightarrow \left(\sum \frac{a}{\sqrt[n]{a^2+3ab+3b^2+2bc}}\right)^n \geq \left(\sum a\right)^{n-2},$$ and we we are done.

PP.21008. If $a,b,c>0$, then $\sum \frac{a}{\sqrt[3]{a+2b}} \geq \left(\sum a\right)^{\frac{2}{3}}$.

Solution. We have $\sum a(a+2b)=\left(\sum a\right)^2$, and by Hölder's inequality we obtain:

$$\left(\sum \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum a(a+2b)\right)\geq \left(\sum a\right)^4$$

$$\Leftrightarrow \left(\sum \frac{a}{\sqrt[3]{a+2b}}\right)^3 \geq \left(\sum a\right)^2 \Leftrightarrow \sum \frac{a}{\sqrt[3]{a+2b}} \geq \left(\sum a\right)^{\frac{2}{3}},$$ and the proof is complete.

PP.21010. Solve in R the following system:

$$\begin{cases} 11(x^2+y^2)+4xy+9yz+zx=(3x+2y+z)^2 \\ 11(y^2+z^2)+4yz+9zx+xy=(3y+2z+x)^2 \\ 11(z^2+x^2)+4z+9xy+yz=(3z+2x+y)^2 \end{cases}.$$

Solution. Adding the equations of the system we get:

$$8(x^2+y^2+z^2-xy-yz-zx)=0.$$

Yields that the solutions of the system are (a,a,a), with $a\in R$.

PP.21063. In all triangle ABC holds $\left(\frac{4R+r}{s}\right)\leq 1+\frac{s^2}{8r^2}$.

Solution. By the item 5.5 from Bottema we have: $9r(4R+r)\leq 3s^2$, i.e.

$\frac{4R+r}{s} \leq \frac{s}{3r}$. It remains to show that:

$\frac{s^2}{9r^2} \leq 1+\frac{s^2}{8r^2} \Leftrightarrow 0 \leq 1+\frac{s^2}{72r^2}$, true and we are done.

PP.21070. Solve in R the following system:

$$\begin{cases} x^3+y^3+z^3+3xyz = yz(y+z)+zt(z+t)+ty(t+y) \\ y^3+z^3+t^3+3yzt = zt(z+t)+tx(t+x)+xz(x+z) \\ z^3+t^3+x^3+3ztx = tx(t+x)+xy(x+y)+yt(y+t) \\ t^3+x^3+y^3+3txy = xy(x+y)+yz(y+z)+zx(z+x) \end{cases}.$$

Solution. Addind up the equations of the system we obtain:

$$\left(x^3+y^3+z^3+3xyz-xy(x+y)-yz(y+z)-zx(z+x)\right)+$$
$$+\left(y^3+z^3+t^3+3yzt-yz(y+z)-zt(z+t)-ty(t+y)\right)+$$
$$+\left(z^3+t^3+x^3+3ztx-zt(z+t)-tx(t+x)-xz(x+z)\right)+$$
$$+\left(t^3+x^3+y^3+3txy-tx(t+x)-xy(x+y)-yt(y+t)\right)=0.$$

By Schur' inequality, all brackets above are ≥ 0. Therefore we have $x=y=z=t$ or if $x=y=z, t=0$ yields by the first equation of the system that $6x^3=2x^3$, which is impossible. In conclusion, the only solution is (a,a,a,a) with $a \in R$, and we are done.

PP.21121. Prove that

$$\{(x,y)\in Z\times Z \mid 2xy+3x+y+2=0\}=\{(x,y)\in Z\times Z \mid 2xy+x+3y+8=0\}.$$

Solution. The statement is not true, for e.g. if $(x,y)=(-8,0)$ we have

$$2xy+3x+y+2=-22 \text{ and } 2xy+x+3y+8=0, \text{ and we are done.}$$

PP.21129. If $a,b,c>0$ then $\sum\sqrt{a^2+b^2}+3\sum\sqrt{a^2-ab\sqrt{3}+b^2} \geq \sqrt{2}(a+b+c)$.

Solution. We have:

$$\sum\sqrt{a^2+b^2}=\sqrt{2}\sum\sqrt{\frac{a^2+b^2}{2}} \geq \sqrt{2}\sum\frac{a+b}{2}=\sqrt{2}(a+b+c),$$ which is stronger that given inequality, and we are done.

PP.21133. Prove that for all $n \in N^*$, the equation $x^2+y^2+z^2=14^n$ has integer solutions.

Solution. If $n=2t$, then $14^{2t}=4^t\cdot(7)^{2t}=4^t(8-1)^{2t}=4^t(8k+1)$.
If $n=2t+1$, then $14^{2t+1}=4^t\cdot(2\cdot 7^{2t+1})=4^t\left(2(8-1)^{2t+1}\right)=4^t(8k-2)=4^t(8k'+6)$.

Therefore, by the theorem of three squares (see the solution of PP.21440), the number 14^n can be written as a sum of three integers squares. The proof is complete.

PP.21142. Prove that

$$\{3a+1 | a\in Z\}\cap\{5b+3 | b\in Z\}\cap\{7c+4 | c\in Z\}=\{105d-17 | d\in Z\}.$$

Solution. A particular solution of the equation $3a+1=5b+3 \Leftrightarrow 3a-5b=2$ is $a=4, b=2$. General solution is $a=5k+4, b=3k+2$ where $k\in Z$.
Yields that $\{3a+1 | a\in Z\}\cap\{5b+3 | b\in Z\}=\{15k+13 | k\in Z\}$.
We proceed analogously with the equation $15k+13=7c+4 \Leftrightarrow 7c-15k=9$, and we obtain the particular solution $c=27, k=12$ and then $c=15d+27, k=7d+12$.
Yields $7c+4=105d+189+4$, i.e. $7c+4=105d'-17$ and we are done.

PP.21145. If $B, A_k\in P(M)$ $(k=1,2,...,n)$, then:

1) $B-\left(\bigcap_{k=1}^{n}A_k\right)=\bigcup_{k=1}^{n}(B-A_k)$;

2) $B-\left(\bigcup_{k=1}^{n}A_k\right)=\bigcap_{k=1}^{n}(B-A_k)$.

Solution. We have that:

1) $x\in B-\left(\bigcap_{k=1}^{n}A_k\right)\Leftrightarrow x\in B$ and $x\notin\bigcap_{k=1}^{n}A_k \Leftrightarrow x\in B$ and exists k such that $x\notin A_k$

$\Leftrightarrow x\in B-A_k \Leftrightarrow x\in\bigcup_{k=1}^{n}(B-A_k)$.

2) $x\in B-\left(\bigcup_{k=1}^{n}A_k\right)\Leftrightarrow x\in B$ and $x\notin\left(\bigcup_{k=1}^{n}A_k\right)\Leftrightarrow x\in B$ and for any k, $x\notin A_k$

$\Leftrightarrow$ for any k, $x\in B-A_k \Leftrightarrow x\in\bigcap_{k=1}^{n}(B-A_k)$, and the proof is complete.

PP.21146. If $a,b,c\in C$, then determine all $n\in N$ such that:

$$(a+b+c)^n+(-a+b+c)^n+(a-b+c)^n+(a+b-c)^n=2^n(a^n+b^n+c^n).$$

Solution. For $a=b=c=1$, we have $3^n+3=3\cdot 2^n \Leftrightarrow 3^{n-1}=2^n-1$, which is true for $n=1, n=2$. If $n\geq 3$, by mathematical induction esily yields that $3^{n-1}>2^n-1$.
So, it remains to verify the relation from the statement for $n\in\{1,2\}$.
(i)For $n=1$; $a+b+c-a+b+c+a-b+c+a+b-c=2(a+b+c)$, true.
(ii)For $n=2$; $(a+b+c)^2+(-a+b+c)^2+(a-b+c)^2+(a+b-c)^2=2^2(a^2+b^2+c^2)$, true. In conclusion we obtain $n\in\{1,2\}$, and we are done.

PP.21147. If $a,b,c,d \in C$, then determine all $n \in N$ such that:

$$(-a+b+c+d)^n+(a-b+c+d)^n+(a+b-c+d)^n+(a+b+c-d)^n=$$
$$=2^n(a^n+b^n+c^n+d^n).$$

Solution. For $d=0$ we obtain the relation from PP.21146, so we must to verify the relation from the statement only for $n=1$ and $n=2$, which easily follows is true.

PP.21148. In all triangle ABC holds:

1) $\sum\left(1-\frac{r}{2R}-2\sin^2\frac{A}{2}\right)^2=\frac{12R^2+4Rr+r^2-2s^2}{4R^2}$;

2) $s\sqrt{3}\le 4R+r$.

Solution. We have the formulas:

$ab+bc+ca=s^2+r^2+4Rr$; $a^2+b^2+c^2=2(s^2-r^2-4Rr)$,

$a^3+b^3+c^3-3abc=(a+b+c)\left[(a+b+c)^2-3(ab+bc+ca)\right]=2s(s^2-3r^2-12Rr)$;

$\cos A+\cos B+\cos C=\frac{R+r}{R}$; $1-2\sin^2\frac{A}{2}=\cos A$.

Therefore,

$$\sum\left(1-\frac{r}{2R}-2\sin^2\frac{A}{2}\right)^2=\sum\left(\cos A-\frac{r}{2R}\right)^2=\sum\cos^2 A-\frac{r}{R}\sum\cos A+\frac{3r^2}{4R^2}=$$
$$=3-\sum\sin^2 A-\frac{r(R+r)}{R^2}+\frac{3r^2}{4R^2}=\frac{12R^2-a^2-b^2-c^2-4Rr-4r^2+3r^2}{4R^2}=$$
$$=\frac{12R^2-2s^2+2r^2+8Rr-4Rr-4r^2+3r^2}{4R^2}=\frac{12R^2+4Rr+r^2-2s^2}{4R^2}.$$

2) The inequality $s\sqrt{3}\le 4R+r$ is the item 5.5. from Bottema.

The proof is complete.

PP.21149. In all triangle ABC holds:

1) $\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)^2=\frac{(4R+r)^2-2s^2}{4R^2}$;

2) $s\sqrt{3}\le 4R+r$.

Solution 1. We use the formulas from PP.21148.

Because $\cos^2\frac{A}{2}=1+\cos A$, we obtain:

$$\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)^2=\sum\left(1-\cos A+\frac{r}{2R}\right)=3+\sum\cos^2 A+\frac{3r^2}{4R^2}-2\sum\cos A+$$
$$+2\sum\frac{r}{2R}-\frac{r}{R}\sum\cos A=6-\sum\sin^2 A+\frac{3r^2}{4R^2}-\frac{2(R+r)}{R}+\frac{3r}{R}-\frac{r}{R}\cdot\frac{R+r}{R}=$$

$$=\frac{24R^2-2(s^2-r^2-4Rr)+3r^2-8R^2-8Rr+12Rr-4Rr-4r^2}{4R^2}=$$
$$=\frac{16R^2+r^2+8Rr-2s^2}{4R^2}=\frac{(4R+r)^2-2s^2}{4R^2}.$$

2) The inequality $s\sqrt{3}\le 4R+r$ is the item 5.5. from Bottema.
The proof is complete.

Solution 2. We use the following:
If $x,y,z\in R$ then:
(*) $(-x+y+z)^2+(x-y+z)^2+(x+y-z)^2+(x+y+z)^2=4(x^2+y^2+z^2)$.
We take in (*) $x=\cos^2\frac{A}{2}, y=\cos^2\frac{B}{2}, z=\cos^2\frac{C}{2}$, and after some algebra we deduce (1). By Bergström's inequality we have:
$\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)^2\ge\frac{1}{3}\left(\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)\right)^2=\frac{(4R+r)^2}{12R^2}$, and by (1) yields (2).

PP.21150. In all triangle ABC holds:
1) $\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)^2=\frac{s^2-2r^2-8Rr}{s^2r^2}$;
2) $s^2\ge 3r^2+12Rr$.

Solution 1. We take in (*) from solution 2 of PP.21149: $x=\frac{1}{h_a}, y=\frac{1}{h_b}, z=\frac{1}{h_c}$, and we deduce (1). By Bergström's inequality we have: $\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)^2\ge\frac{1}{3}\left(\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)\right)^2=\frac{1}{3r^2}$, and from (1) we obtain (2), and we are done.

Solution 2. Using the item 5.8 from Bottema $(s^2\ge 16Rr-5r^2)$ it suffices to prove that:
$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow 4Rr\ge 8r^2\Leftrightarrow R\ge 2r$, i.e. well-known Euler's inequality.

2.Other solutions from some problems from SSMJ

By Nela Ciceu, Roşiori, Bacău, Romania
and
Roxana Mihaela Stanciu, Buzău, Romania

- **5313:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles if they both have perimeter 256 and area 1008.

Solution:

Let $2a, 2b, 2b$, with $a < 2b$ the sides of triangle.
We have

$$a + 2b = 128 \text{ and } a\sqrt{4b^2 - a^2} = 1008.$$

Since

$$2b = 128 - a \text{, yields } 4b^2 - a^2 = (2b + a)(2b - a) = 256(64 - a)$$

then

$$a\sqrt{4b^2 - a^2} = 1008 \Leftrightarrow 16a\sqrt{64 - a} = 1008 \Leftrightarrow a\sqrt{64 - a} = 63$$

and denoting

$$64 - a = x^2$$

we obtain the equation

$$(64 - x^2)x = 63 \Leftrightarrow x^3 - 64x + 63 = 0 \Leftrightarrow (x - 1)(x^2 + x - 63) = 0.$$

- For $x = 1$ yields the triangle with the sides 126, 65, 65.
- For $x = \frac{-1 + \sqrt{253}}{2}$ (the value $x = \frac{-1 - \sqrt{253}}{2}$ it is not possible)yields the triangle with the sides $1 + \sqrt{253}, \frac{255 - \sqrt{253}}{2}, \frac{255 - \sqrt{253}}{2}$.

- **5314:** *Proposed by Roger Izard, Dallas TX*

  A biker and a hiker like to workout together by going back and forth on a road which is ten miles long. One day, at 8 AM, at the starting end of the road, they went out together. The biker soon got far past the hiker, reached the end of the road, reversed his direction, and soon passed by the hiker at 9:06 AM. Then, the biker got down to the beginning part of the road, reversed his direction, and got back to the hiker at 9:24 AM. The biker and the hiker were, then, going in the same direction. Calculate in miles per hour the speeds of the hiker and the biker.

Solution:

Let b biker's speed and h hiker's speed. We use the fact that $distance = speed \times time$.
As from 8 AM to 9:06 AM we have 1.1 hours, and from 9:06 AM to 9:24 AM we have 0.3 hours we have the equations:

$$1.1b + 1.1h = 20,\ 0.3b = 1.1h + 1.1h + 0.3h.$$

We obtain $b = \frac{1250}{77}$ and $h = \frac{150}{77}$.

- **5315:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The hexagonal numbers have the form $H_n = 2n^2 - n$, $n = 1, 2, 3, \ldots$. Prove that infinitely many hexagonal numbers are the sum of two hexagonal numbers.

Solution:

We want to find a, b, c such that $H_a = H_b + H_c$.
We choose $a = b + 1$ and we get

$$2(b+1)^2 - (b+1) = 2b^2 - b + 2c^2 - c \Leftrightarrow 4b + 1 = 2c^2 - c.$$

If we take $c = 4n + 1$, yields that $b = 8n^2 + 3n$ and is easily to verify that

$$H_{8n^2+3n+1} = H_{8n^2+3n} + H_{4n+1},$$

so there is infinitely many hexagonal numbers which are the sum of two hexagonal numbers.

- **5316:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $\{u_n\}_{n \geq 0}$ be a sequence defined recursively by

$$u_{n+1} = \sqrt{\frac{u_n^2 + u_{n-1}^2}{2}}.$$

  Determine $\lim\limits_{n\to\infty} u_n$ in terms of u_0, u_1.

Solution:

We note that the given sequence has positive terms.

The equation $2x^2 - x - 1 = 0$ has the roots 1 and $-\frac{1}{2}$.

After some algebra, we obtain

$$u_n^2 = \frac{u_0^2 + 2u_1^2}{3} + \frac{2(-1)^n (u_0^2 - u_1^2)}{3 \cdot 2^n}.$$

Hence,

$$\lim_{n\to\infty} u_n = \sqrt{\frac{u_0^2 + 2u_1^2}{3}}.$$

- **5317:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_k, b_k > 0,\ 1 \le k \le n$, be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

$$\frac{1}{n^3}\left(\sum_{k=1}^{n} b_k\right)^5 \le \sum_{k=1}^{n} \frac{b_k^5}{a_k}.$$

Solution:

Applying Hölder's inequality we obtain

$$\left(\sum_{k=1}^{n} \frac{b_k^5}{a_k}\right)\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} 1\right)\left(\sum_{k=1}^{n} 1\right)\left(\sum_{k=1}^{n} 1\right) \ge \left(\sum_{k=1}^{n} \sqrt[5]{\frac{b_k^5}{a_k} \cdot a_k \cdot 1 \cdot 1 \cdot 1}\right)^5,$$

from where easily yields the given inequality.

3.Dreapta lui Gauss

Prof. Alexandru Elena-Marcela, gradul didactic II
Școala Gimnazială nr. 3 Baia, structura Bogata

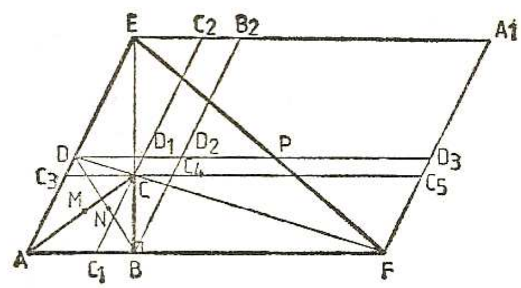
Definiție: Fie patrulaterul convex *ABCD*. Dacă $AB \cap CD = \{E\}$ și $BC \cap AD = \{F\}$ atunci *ABCDEF* se numește *patrulater complet*, iar segmentele $[AC]$, $[BD]$ și $[EF]$ se numesc *diagonalele* patrulaterului complet.

Mijloacele diagonalelor unui patrulater complet sunt coliniare. ***(Dreapta lui Gauss)***

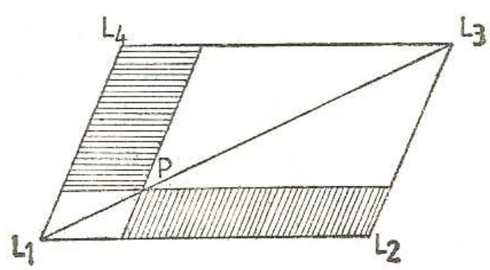
Demonstrație:

Prin punctele *E*, *F*, *B*, *C* și *D* se construiesc paralelele la laturile opuse și se consideră notațiile din figura alăturată.

Fie omotetia de centru *A* și raport 2. Atunci pentru a arăta că *M*, *N*, *P* sunt coliniare revine la a arăta că punctele *C*, D_2 și A_1 sunt coliniare.



Se știe că: *Într-un paralelogram o condiție necesară și suficientă pentru ca un punct să aparțină diagonalei este ca paralelogramele determinate de punct și vârfurile ce nu aparțin diagonalei să aibă ariile egale.*



Astfel se demonstrează că $A_{C_2D_1D_2B_2} = A_{C_4D_2D_3C_5}$ sau $A_{C_2CC_4B_2} = A_{CD_1D_3C_5}$.

Din $C \in EB$ rezultă că $A_{C_2CC_4B_2} = A_{C_3CC_1A}$, iar din $C \in DF$ rezultă că $A_{CD_1D_3C_5} = A_{C_3CC_1A}$, ceea ce trebuia de demonstrat.

Bibliografie:

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