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1. The inequality Ionescu - Weitzenböck

by D. M. BĂTINEŢU-GIURGIU[1] and NECULAI STANCIU[2]

In memory of Ion Ionescu

"Give to Caesar what belongs to Caesar and to God what belongs to God."
Matthew 22:21

ION IONESCU (BIZET)
(1870 – 1946)

The aim of this note is to proof that the *Weitzenböck*'s inequality must be named the *Ionescu-Weitzenböck*'s inequality.

In Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 2 , 15 October 1897, on page 52, *Ion Ionescu*, the founder of Romanian Mathematical Gazette, published the problem:

***273.** Prove that there is no triangle for which the inequality:

$$4S\sqrt{3} > a^2 + b^2 + c^2$$

be satisfied. **(I. IONESCU.)**

The solution of the problem 273, apperead in Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 12 , 15 August 1898, on pages 281, 282 and 283, as follows:

***Problema 273**. - *Prove that there is no triangle for which the inequality:*

$$4S\sqrt{3} > a^2 + b^2 + c^2$$

be satisfied. **(I. IONESCU.)**

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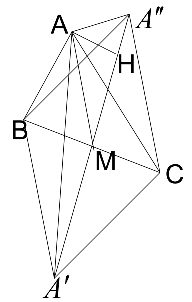
Solution by D-l ***N. G. Muzicescu*** (Student, Iaşi).



Let ABC be a triangle, and we construct around the side BC the equailateral triangles BCA' and BCA''; so $M = A'A'' \cap BC$ will be the middle which of these two lines.
Let $AA' = d$ and $AA'' = d'$.
In triangle $AA'A''$, applying well known theorems we deduce that:

(1) $d^2 + d'^2 = 2 \cdot AM^2 + 2 \cdot A''M^2$
(2) $d^2 - d'^2 = 4 \cdot A''M^2 \cdot MH$;
and from triangle ABC, yields:
(3) $b^2 + c^2 = 2 \cdot AM^2 + \frac{a^2}{2}$.
On the oher hand because $A''M$ is the height of the equilateral triangle with the lengh of the side a we have:
(4) $A''M = \frac{1}{2} a\sqrt{3}$.
By the rtfelations (1), (2), (3) and (4) we deduce:
(5) $d^2 + d'^2 = a^2 + b^2 + c^2$,
(6) $d^2 - d'^2 = 2a \cdot MH\sqrt{3}$,
but: $2a \cdot MH$ reprezint 4 times the area of the triangle ABC, because MH is evidently is equal with the height of this triangle; making the substitution by (6) and (5), we have: $2d'^2 = a^2 + b^2 + c^2 - 4S\sqrt{3}$ which yields:
$a^2 + b^2 + c^2 \geq 4S\sqrt{3}$,
So the given inequality is impossible.

Solution by: ***I. Moscuna*** (Bucharest) and ***I. Penescu*** (Bucharest).
We have:
$S = \frac{1}{2} bc \sin A$, $a^2 = b^2 + c^2 - 2bc\cos A$, $\sqrt{3} = \operatorname{cotg} 30^0 = \frac{\cos 30^0}{\sin 30^0}$.
Replacing , simplifying with 2, and letting in the member II only $b^2 + c^2$ we have: $bc\left(\sin A \cdot \frac{\cos 30^0}{\sin 30^0} + \cos A\right) > b^2 + c^2$, which yields:

(1) $$2bc\sin(A+30^0) > b^2+c^2 \ .$$

On the other hand we have:

$(b-c)^2 \geq 0$, so:

(2) $$2bc \leq b^2+c^2$$

The inequality (1) is satisfied for $A<150^0$, because otherwise the member I would be negative and could not be greater like the member II which is positive. Assuming this condition satisfied we can divide (1) by (2). Therefore:

(3) $$\sin(A+30^0) > 1.$$

This inequalty is impossible, the sine can not be greater than unity. So the inequality (1) and therefore the given inequality absurd for any triangle. If (3) and (2) becomes equalities, then (1) becomes equality. So must have $A+30^0 = 90^0$, i.e. $A=60^0$ and $b=c$, i.e. the triangle must be equilateral. Hence the given inequality becomes equality for any equilateral triangle.

Solution by Miss ***Maria Rugescu*** (Student, Iaşi) and by *Mrs.* : ***Th. M. Vladimirescu*** (Rîmnicul Vîlcei); ***G. G. Urechiă*** and ***I. Sichitiu*** (Sc. Sp. De Art. şi Geniu) and ***Corneliu P. Ionescu*** (Student Cl. VI Lic. Galaţi).

The given inequality becomes successively:

$4\sqrt{p(p-a)(p-b)(p-c)}\cdot\sqrt{3} > a^2+b^2+c^2$,

$16\cdot p(p-a)(p-b)(p-c)\cdot 3 > (a^2+b^2+c^2)^2$,

$3\cdot 2p(2p-2a)(2p-2b)(2p-2c) > (a^2+b^2+c^2)^2$,

$3\cdot(a+b+c)(b+c-a)(a+c-b)(a+b-c) > (a^2+b^2+c^2)^2$,

$3\cdot[(b+c)^2-a^2][a^2-(b-c)^2] > (a^2+b^2+c^2)^2$,

$3\cdot[2bc+c^2+b^2-a^2][2bc+a^2-b^2-c^2] > (a^2+b^2+c^2)^2$,

$3\cdot[4b^2c^2-(a^2-b^2-c^2)^2] > (a^2+b^2+c^2)^2$,

$3\cdot[2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4] > a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2$,

$4a^2b^2+4a^2c^2+4b^2c^2-4a^4-4b^4-4c^4 > 0$,

$2[2a^4+2b^4+2c^4-2a^2b^2-2a^2c^2-2b^2c^2] < 0$,

$2[(a^2-b^2)^2+(a^2-c^2)^2+(b^2-c^2)^2] < 0$.

But the last inequality is imposible, becase all the terms from the member I ar positive.
Hence, the given inequality is imposible in all triangles.
The last inequality becomes equality if and only if $a=b=c$, i.e. when the triangle is equailateral.

Also solved the above problem in different ways and by Mrs.: *A. Iliovici, I. Nicolaescu, Emil G. Niţescu, V. V. Cambureanu* and *C. Vintilă*.

In the year 1919, *Roland Weitzenböck* publishead in Mathematische Zeitschrift, Vol. 5, No. 1-2, pp. 137-146 the article *Uber eine Ungleichung in der Dreiecksgeometrie*, where he proof that:

In any triangle ABC, with usual notations holds the inequality:

$$a^2+b^2+c^2 \geq 4\sqrt{3}S$$

We observe that the inequality of *Ion Ionescu* is the same with the inequality of *Weitzenböck*, and therefore from this moment the inequality of *Weitzenböck* must be named the inequality ***Ionescu-Weitzenböck***. The inequality ***Ionescu – Weitzenböck***, was given to solve at third *IMO, Veszprém, Ungaria, 8-15 iulie 1961*.
A number of 11 proofs of the ***Ionescu-Weitzenböck***'s inequality was presented by *Arthur Engel* in the book *Problem solving strategies*, Springer Verlag, 1998.

We discovered the above while we working on an article about *Weitzenböck*'s inequality. Also we also published 23 demonstrations (see Romanian Mathematical Gazette – No. 1/2013, pp. 1-10) and 10 generalizations (see the math journal – Journal of Science and Arts – No. 1/2013) other than those published of the ***Ionescu-Weitzenböck***'s inequality.

TREI INEGALITĂŢI - MAI MULTE METODE

Prof. Aurora Olivia Mironescu

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1. Dacă $a,b,c>0$, **atunci:** $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\geq\dfrac{3}{2}$

şi să se arate că dacă $a_1,a_2,...,a_n$ **sunt strict pozitive, atunci**

$$\frac{a_1}{a_1+a_2+...+a_n}+\frac{a_2}{a_1+a_2+...+a_n}+...+\frac{a_n}{a_1+a_2+...+a_n}\geq\frac{n}{n-1}.$$

(Olimpiada de matematică, Polonia,1959)

Soluţie:

Metoda I: Ţinând cont de inegalitatea mediilor, rezultă:

$$\left(\sum(a+b)\right)\cdot\left(\sum\frac{1}{a+b}\right)\geq 9 \Leftrightarrow 2\left(\sum a\right)\cdot\left(\sum\frac{1}{a+b}\right)\geq 9 \Leftrightarrow (a+b+c)\cdot\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq\frac{9}{2}\Leftrightarrow$$

$$\Leftrightarrow 1+\frac{c}{a+b}+1+\frac{a}{b+c}+1+\frac{b}{c+a}\geq\frac{9}{2}\Leftrightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Metoda II:

În inegalitatea Cauchy - Buniakovski:

$$\left(x_1^2+x_2^2+x_3^2\right)\left(y_1^2+y_2^2+y_3^2\right)\geq\left(x_1y_1+x_2y_2+x_3y_3\right)^2,$$

punând

$$x_1=\sqrt{\frac{a}{b+c}},\quad x_2=\sqrt{\frac{b}{c+a}},\quad x_1=\sqrt{\frac{c}{a+b}},$$

$$y_1=\sqrt{a(b+c)},\quad y_2=\sqrt{b(c+a)},\quad y_1=\sqrt{c(a+b)},$$

obţinem:

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)(ab+ac+bc+ba+ca+cb)\geq(a+b+c)^2,$$

de unde

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq\frac{a^2+b^2+c^2+2ab+2bc+2ca}{2ab+2bc+2ca}\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}.$$

Metoda III:

Notând $b+c=x,\ \ c+a=y,\ \ a+b=z,$ deducem că $a+b+c=\dfrac{x+y+z}{2}$,

de unde: $a=\dfrac{-x+y+z}{2},\ \ b=\dfrac{x-y+z}{2},\ \ c=\dfrac{x+y-z}{2}$ cu $x,y,z>0$.

Înlocuind în relaţia din enunţ, avem:

$$\frac{-x+y+z}{2x}+\frac{x-y+z}{2y}+\frac{x+y-z}{2z}\geq\frac{3}{2}\Leftrightarrow\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)\geq 6.$$

Ultima inegalitate este adevărată deoarece $\dfrac{x}{y}+\dfrac{y}{x}\geq 2,\ \ \dfrac{y}{z}+\dfrac{z}{y}\geq 2,\ \ \dfrac{x}{z}+\dfrac{z}{x}\geq 2$, simple consecinţe ale inegalităţii mediilor.

Metoda IV:

Se consideră tripletele (a,b,c), $\left(\frac{1}{b+c},\frac{1}{a+c},\frac{1}{a+b}\right)$ de aceeaşi monotonie .

Atunci $\max S=\sum\frac{a}{b+c}$, adică exact membrul stâng al inegalităţii.

Pe de altă parte au loc inegalităţile:

$$\sum\frac{a}{b+c}\geq a\frac{1}{a+c}+b\frac{1}{a+b}+c\frac{1}{b+c}$$
$$\sum\frac{a}{b+c}\geq a\frac{1}{a+b}+c\frac{1}{a+c}+b\frac{1}{b+c},$$

care însumate dau inegalitatea cerută.

Metoda V:

Presupunem fără a reduce generalitatea că $a+b+c=1$.

Dacă $a+b+c=s$ împărţind prin s rezultă $\frac{a}{s}+\frac{b}{s}+\frac{c}{s}=1$ şi înlocuind în relaţia de demonstrat aceasta devine

$$\frac{\frac{a}{s}}{\frac{b}{s}+\frac{c}{s}}+\frac{\frac{b}{s}}{\frac{c}{s}+\frac{a}{s}}+\frac{\frac{c}{s}}{\frac{a}{s}+\frac{b}{s}}\geq\frac{3}{2},$$

care pentru

$$\frac{a}{s}=a_1,\frac{b}{s}=b_1,\frac{c}{s}=c_1 \text{ cu } a_1+b_1+c_1=1$$

se transformă în

$$\frac{a_1}{b_1+c_1}+\frac{b_1}{c_1+a_1}+\frac{c_1}{a_1+b_1}\geq\frac{3}{2}.$$

Inegalitatea de demonstrat este echivalentă cu

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}\geq\frac{3}{2}.$$

Se demonstrează că are loc inegalitatea $\frac{x}{1-x}>\frac{9x-1}{4}$, $\forall x\in(0,1)$.

Considerăm funcţia $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{x}{1-x}$ cu derivata $f'(x)=\frac{1}{(1-x)^2}$ care

dezvoltată în serie Taylor devine:

$$f(x)=f(x_0)+\frac{1}{1!}(x-x_0)f'(x_0)+\frac{1}{2!}(x-x_0)^2 f''(x_0)+\ldots\Rightarrow$$

$$f(x)\geq f(x_0)+(x-x_0)f'(x_0).$$

Pentru $x_0=\frac{1}{3}$

$$f(x_0)+(x-x_0)f'(x_0)=f\left(\frac{1}{3}\right)+\left(x-\frac{1}{3}\right)f'\left(\frac{1}{3}\right)=\left(x-\frac{1}{3}\right)\frac{9}{4}+\frac{1}{2}=\frac{9x-1}{4}\Rightarrow$$

$$\Rightarrow f(x)\geq\frac{9x-1}{4}\Rightarrow\frac{x}{1-x}\geq\frac{9x-1}{4}.$$

Deci

$$\left.\begin{array}{l}\frac{a}{1-a}\geq\frac{9a-1}{4}\\ \frac{b}{1-b}\geq\frac{9b-1}{4}\\ \frac{c}{1-c}\geq\frac{9c-1}{4}\end{array}\right|\overset{(+)}{\Rightarrow}\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}\geq\frac{9a-1}{4}+\frac{9b-1}{4}+\frac{9c-1}{4}=\frac{9(a+b+c)-3}{4}=\frac{3}{2}\Rightarrow$$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Pentru a demonstra inegalitatea generală se fac notaţiile:

$$a_2+a_3+\ldots+a_n=(n-1)A_1,\quad a_1+a_3+\ldots+a_n=(n-1)A_2,\quad \ldots,\quad a_1+a_2+\ldots+a_{n-1}=(n-1)A_n$$

.

Deci

$$a_1=(2-n)A_1+A_2+\ldots+A_n,\ldots,a_n=A_1+A_2+\ldots+(2-n)A_n.$$

Inegalitatea de demonstrat devine:

$$\frac{(2-n)A_1+A_2+\ldots+A_n}{(n-1)A_1}+\ldots+\frac{A_1+A_2+\ldots+(2-n)A_n}{(n-1)A_n}\geq\frac{n}{n-1}$$

sau

$$\left(\frac{A_3}{A_1}+\ldots+\frac{A_n}{A_1}\right)+\ldots+\left(\frac{A_1}{A_n}+\ldots+\frac{A_{n-1}}{A_n}\right)\geq n(n-1),$$

care este evidentă, deoarece $\frac{a}{b}+\frac{b}{a}\geq 2$ şi numărul fracţiilor din membrul stâng al inegalităţii este $2C_n^2$.

2.Fie numerele a_1,a_2,a_3,a_4,a_5 care verifică inegalitatea

$$(a_1+a_2+a_3+a_4-a_5)^2\geq 3\left(a_1^2+a_2^2+a_3^2+a_4^2-a_5^2\right).$$

Să se demonstreze că oricare ar fi $x,y\in\mathbb{R}$, are loc inegalitatea:

$$(a_1+a_2+a_3+a_4-a_5-x-y)^2\geq a_1^2+a_2^2+a_3^2+a_4^2-a_5^2-x^2-y^2.$$

(O.N.M-1983)

Soluţie:

Inegalitatea din enunţ este echivalentă cu

$$(a_1+a_2+a_3+a_4-a_5-x-y)^2+x^2+y^2\geq a_1^2+a_2^2+a_3^2+a_4^2-a_5^2$$

sau înmulţind-o cu 3

$$3\left((a_1+a_2+a_3+a_4-a_5-x-y)^2+x^2+y^2\right)\geq 3\left(a_1^2+a_2^2+a_3^2+a_4^2-a_5^2\right) \quad (1)$$

Pentru a stabili (1) se demonstrează că

$$3\left((a_1+a_2+a_3+a_4-a_5-x-y)^2+x^2+y^2\right)\geq (a_1+a_2+a_3+a_4-a_5)^2 \quad (2)$$

Şi apoi se foloseşte inegalitatea din enunţ.

Se notează $a_1+a_2+a_3+a_4-a_5=\alpha$, inegalitatea (2) este echivalentă cu

$$3\left((\alpha-x-y)^2+x^2+y^2\right)\geq\alpha^2 \text{ sau } \frac{(\alpha-x-y)^2+x^2+y^2}{3}\geq\left(\frac{(\alpha-x-y)+x+y}{3}\right)^2 \text{ care}$$

este tocmai inegalitatea lui Jensen aplicată funcţiei $f:\mathbb{R}\to\mathbb{R},\ f(t)=t^2$ şi numerelor $\alpha-x-y,x,y$. Cu aceasta inegalitatea este demonstrată.

3. Fiind date numerele reale pozitive distincte $a_1,a_2,\ldots,a_n$, $n\in\mathbb{N}^*$ şi o rearanjare $b_1,b_2,\ldots,b_n$ a lor să se demonstreze inegalitatea:

$$\left(a_1^2+b_1\right)\left(a_2^2+b_2\right)\ldots\left(a_n^2+b_n\right)\geq\left(a_1^2+a_1\right)\left(a_2^2+a_2\right)\ldots\left(a_n^2+a_n\right).$$

(O.N.M 2009)

Soluţie:

Metoda I:

Dacă vreunul dintre $a_i=0$ concluzia este evidentă.

Fără a restrânge generalitatea putem presupune că $0<a_1<a_2<\ldots<a_n$.

Dintre toate rearanjările $x_1,x_2,\ldots,x_n$ ale numerelor $a_1,a_2,\ldots,a_n$ se alege una astfel încât produsul $\left(a_1^2+x_1\right)\left(a_2^2+x_2\right)\ldots\left(a_n^2+x_n\right)$ să aibă valoarea minimă.

Fie $c_1,c_2,\ldots,c_n$ o astfel de rearanjare şi $i<j$ doi indici oarecare.

Produsul corespunzător rearanjării $c_1,c_2,\ldots c_i,\ldots,c_j,\ldots,c_n$ este mai mic decât cel corespunzător rearanjării $c_1,c_2,\ldots c_j,\ldots,c_i,\ldots,c_n$.

Rezultă că $\left(a_i^2+c_i\right)\left(a_j^2+c_j\right)\geq\left(a_i^2+c_j\right)\left(a_j^2+c_i\right)$ adică $\left(a_i^2-a_j^2\right)\left(c_j-c_i\right)\leq 0$ de unde se deduce că $c_i<c_j$. Deci $c_k=a_k$ oricare ar fi indicele k şi se obţine inegalitatea din enunţ.

Metoda II:

Pornind de la inegalitatea $x^2+y\geq\dfrac{y(x+1)}{y+1}$ adevărată pentru orice numere x,y reale şi pozitive, fiind echivalentă cu $(x-y)^2\geq 0$ înmulţind inegalităţile corespunzătoare perechilor $\left(a_i,b_i\right)$ cu $1\leq i\leq n$ se obţine inegalitatea cerută.

Metoda III:

Fără a restrânge generalitatea putem presupune că $0<a_1<a_2<\ldots<a_n$.

Se consideră $b_1=a_{\sigma(1)},b_2=a_{\sigma(2)},\ldots,b_n=a_{\sigma(n)}$ şi fie produsele

$$P_\sigma = \left(a_1^2 + a_{\sigma(1)}\right)\left(a_2^2 + a_{\sigma(2)}\right)\ldots\left(a_n^2 + a_{\sigma(n)}\right).$$

Există o permutare σ astfel încât P_σ să fie minim?

Dacă $\sigma = e$

Fie $i < j$ $\tau_0 = \begin{pmatrix} 1 & 2 & i & \ldots & j & \ldots & n \\ \sigma(1) & \sigma(2) & \sigma(j) & \ldots & \sigma(i) & \ldots & \sigma(n) \end{pmatrix}$.

Presupunem că σ este permutarea care face ca P_σ să fie minim

$$\Leftrightarrow P_\sigma \le P_\tau, (\forall)\tau \in S_n$$

Pentru

$$\tau = \tau_0 \Rightarrow P_\sigma \le P_{\tau_0} \Rightarrow \left(a_i^2 + a_{\sigma(i)}\right)\left(a_j^2 + a_{\sigma(j)}\right) \le \left(a_i^2 + a_{\sigma(j)}\right)\left(a_j^2 + a_{\sigma(i)}\right) \Leftrightarrow$$

$$\Leftrightarrow a_i^2 a_j^2 + a_i^2 a_{\sigma(j)} + a_j^2 a_{\sigma(i)} + a_{\sigma(i)} a_{\sigma(j)} \le a_i^2 a_j^2 + a_i^2 a_{\sigma(i)} + a_j^2 a_{\sigma(j)} + a_{\sigma(i)} a_{\sigma(j)} \Rightarrow$$

$$\Rightarrow a_i^2\left(a_{\sigma(j)} - a_{\sigma(i)}\right) - a_j^2\left(a_{\sigma(j)} - a_{\sigma(i)}\right) \le 0 \Leftrightarrow \left(a_{\sigma(j)} - a_{\sigma(i)}\right)\left(a_i^2 - a_j^2\right) \le 0.$$

Dar

$$i < j \Rightarrow a_i < a_j \Rightarrow a_i^2 - a_j^2 < 0 \Rightarrow a_{\sigma(j)} - a_{\sigma(i)} \ge 0 \Rightarrow \sigma(j) > \sigma(i), (\forall) j > i$$

,

σ nu are nici o inversiune $\Rightarrow \sigma = e$

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3.Câteva aplicaţii ale numerelor complexe în geometrie

Profesor Serban George-Florin
Liceul Tehnologic “Grigore Moisil “ Braila

Reamintim ca pentru orice numar complex z=x+iy i se asociaza punctul M de coordonate (x,y).Punctul M se numeste imaginea geometrica a numarului complex z=x+iy iar z se numeste afixul lui M. In continuare vom lucra cu afixe.
Voi prezenta in continuare cateva formule cu afixe iar apoi exemple la aceste formule.

1)Daca $M_1(z_1)$ si $M_2(z_2)$ atunci $M((z_1+z_2)/2)$ este mijlocul segmentului $[M_1M_2]$.
2)Daca $M_1(z_1)$ si $M_2(z_2)$ atunci $M(z)$ unde $z=(z_1+k z_2)/(1+k)$ unde $MM_1=kMM_2$, punctul M imparte segmentul $[M_1M_2]$ in raportul k.
3) Daca $M_1(z_1)$ si $M_2(z_2)$ atunci $m(\angle M_2OM_1)=\arg(z_2)-\arg(z_1)=\arg(z_2/z_1)$.
4) Daca $M_1(z_1)$, $M_2(z_2)$ si $M_3(z_3)$ sunt coliniare daca $(z_3-z_1)/(z_2-z_1) \in R$.
5) Daca $M_i(z_i)$, $i \in \{1,2,3,4\}$, $M_1M_2 \perp M_3M_4$ daca $(z_1-z_2)/(z_3-z_4) \in iR$.
6) Daca $M_1(z_1)$ si $M_2(z_2)$ atunci $M_1M_2=|z_1-z_2|$.
7)Fie $\Delta M_1M_2M_3 \approx \Delta M_1'M_2'M_3'$ unde $M_i(z_i)$ si $M_i'(z_i')$, $i \in \{1,2,3\}$ atunci $(z_2-z_1)/(z_3-z_1)=(z_2'-z_1')/(z_3'-z_1')$.
8) Daca $M_i(z_i)$, $i \in \{1,2,3,4\}$ atunci $M_1M_2M_3M_4$ este paralelogram daca $z_1+z_3=z_2+z_4$.
9) Fie $\Delta M_1M_2M_3$, $M_i(z_i)$, $i \in \{1,2,3\}$. Afixele unor puncte importante in triunghi G,H,O,I sunt centru de greutate, ortocentrul, centrul cercului circumscris, si centrul cercului inscris $\Delta M_1M_2M_3$.
$Z_G=(z_1+z_2+z_3)/3$, $z_H=z_1+z_2+z_3$, $z_O=0$ si $z_I=(a z_1+b z_2+c z_3)/(a+b+c)$. $a=M_2M_3$, $b=M_1M_3$, $c=M_1M_2$.

Propun ca exercitii sa se demonstreze aceste formule.

In continuare vom rezolva probleme de geometrie (inegalitati , asemanare , coliniaritate , concurenta, minim, congruenta , puncte conciclice , identitati) si anumite teoreme importante si o serie de probleme date la olimpiade si concursuri.

Aplicatii:

1) **Teorema lui Papuss**

Fie ΔABC ,$A^{|} \in BC$, $B^{|} \in AC$, $C^{|} \in BC$, cu $A^{|}B/ A^{|}C= B^{|}C/ B^{|}A= C^{|}A/ C^{|}B=k$.Atunci ΔABC si $\Delta A^{|} B^{|} C^{|}$ au acelasi centru de greutate.
Solutie: Fie $A(z_1)$, $B(z_2)$,$C(z_3)$ $A^{|}B/ A^{|}C=k$ atunci $A^{|}((z_{2+k}\, z_3)/(1+k))$ Analog $B^{|}((z_{3+k}\, z_1)/(1+k))$, $C^{|}((z_{1+k}\, z_2)/(1+k))$.Fie G centrul de greutate al ΔABC , $Z_G=(z_1+z_2+z_3)/3$. Fie $G^{|}$ centrul de greutate al $\Delta A^{|} B^{|} C^{|}$ atunci $Z_{G^{|}}=(z_{A^{|}}+z_{B^{|}}+z_{C^{|}})/3=\Sigma((z_{2+k}\, z_3)/3(1+k))=(z_1+z_2+z_3)/3= Z_G$ deci $G=G^{|}$.

2)Dreapta lui Euler.
Intr-un ΔABC , punctele H, G , O sunt coliniare.
Solutie: $Z_G=(z_1+z_2+z_3)/3$,$z_H= z_1+z_2+z_3$, $z_O=0$, $(z_H - z_O)/(Z_G - z_O) =3 \in R$ deci punctele H, G , O sunt coliniare.

3)O generalizare a teoremei lui Pompeiu.
Daca Δ ABC este un triunghi oarecare si M un punct in planul sau , atunci se poate construi un triunghi cu lungimile laturilor aMA , bMB ,cMC daca si numai daca M nu apartine cercului circumscris Δ ABC.
Solutie: Se arata ca aMA <bMB +cMC si analoagele , adica
$|z_2-z_3||z-z_1| <|z_1-z_3||z-z_2|+|z-z_3||z_1-z_2|$
$|z_2-z_3||z-z_1|=|zz_2-z_1z_2-zz_3+z_1z_3|=|zz_2-zz_1+zz_1-zz_3+z_1z_3-z_2z_3+z_2z_3-z_1z_2|=|(z_2-z_1)(z-z_3)+(z_1-z_3)(z-z_2)|<|z-z_3||z_1-z_2|+|z_1-z_3||z-z_2|$. q e d .

4) Inegalitatea lui Ptolomeu :
In patrulaterul convex ABCD avem inegalitatea AC*BD≤AB*CD+AD*BC.
Solutie: Fie A(0) , $B(z_1)$, $C(z_2)$, $D(z_3)$. Folosim identitatea $z_2(z_1-z_3)+z_1(z_3-z_2)+z_3(z_2-z_1)$, $z_2(z_1-z_3)= z_1(z_2-z_3)+z_3(z_1-z_2)$, $|z_2(z_1-z_3)|=|z_1(z_2-z_3)+z_3(z_1-z_2)|\leq|z_1||(z_2-z_3)|+|z_3|(z_1-z_2)|$ deci AC*BD≤AB*CD+AD*BC.

5) Sa se arate ca intr-un ΔABC au loc formulele :
a) $OI^2=R^2-2Rr$
b) $OH^2=9R^2-(a^2+b^2+c^2)$
c) $3MG^2=MA^2+MB^2+MC^2-(a^2+b^2+c^2)/3$.
Indicatii: Se folosesc formulele
$Z_G=(z_1+z_2+z_3)/3$,$z_H= z_1+z_2+z_3$, $z_O=0$ si $z_I=(a\, z_1+ b\, z_2 +c\, z_3)/(a+b+c)$, $A(z_1)$, $B(z_2)$, $C(z_3)$ si apoi se foloseste formula $|z|^2=z\bar{z}$.
Voi demonstra punctul b) (celelelalte se fac analog).
$OH^2=|z_1+z_2+z_3|^2=|z_1|^2+|z_2|^2+|z_3|^2+\bar{z}_1z_2+\bar{z}_1z_3+\bar{z}_2z_1+\bar{z}_2z_3+\bar{z}_3z_1+\bar{z}_3z_2$.
Calculez $a^2+b^2+c^2=|z_2-z_3|^2+|z_1-z_3|^2+|z_1-z_2|^2=|z_2|^2+|z_3|^2-\bar{z}_2z_3-\bar{z}_3z_2+|z_1|^2+|z_3|^2-\bar{z}_1z_3-\bar{z}_3z_1+|z_1|^2+|z_2|^2-\bar{z}_1z_2-\bar{z}_2z_1=6R^2-(\bar{z}_1z_2+\bar{z}_1z_3+\bar{z}_2z_1+\bar{z}_2z_3+\bar{z}_3z_1+\bar{z}_3z_2)$.
$9R^2-(a^2+b^2+c^2)=3R^2+\bar{z}_1z_2+\bar{z}_1z_3+\bar{z}_2z_1+\bar{z}_2z_3+\bar{z}_3z_1+\bar{z}_3z_2= OH^2$. qed.

6)Daca $A_0A_1...A_{n-1}$ este un poligon regulat cu n laturi inscris in cercul unitate , atunci $|A_0A_1||A_0A_2|....|A_0A_{n-1}|=n$.
Solutie: $A_0(1)$, $A_1(\varepsilon)$ ... $A_{n-1}(\varepsilon^{n-1})$ unde $\varepsilon^n=1$, ε este radacina de ordin n a unitatii.
$X^n-1=(x-1)(x^{n-1}+x^{n-2}+....+1)=(x-1)(x-\varepsilon)....(x-\varepsilon^{n-1})$,
$X=1$, $n=(1-\varepsilon)....(1-\varepsilon^{n-1})$, $n=|(1-\varepsilon)|...|(1-\varepsilon^{n-1})|=|A_0A_1||A_0A_2|....|A_0A_{n-1}|$.

7)Intr-un cerc se inscrie patrulaterul ABCD.Sa se arate ca centrele de greutate G_1, G_2, G_3, G_4 ale triunghiurilor ABC ,BCD ,CDA ,DAB sunt situate pe cerc.

Solutie: Fie $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$, $z_{G1}=(z_1+z_2+z_3)/3$, $z_{G2}=(z_4+z_2+z_3)/3$, $z_{G3}=(z_4+z_1+z_3)/3$, $z_{G4}=(z_4+z_2+z_1)/3$. Ca G_1, G_2, G_3, G_4 ca sa fie conciclice trebuie ca sa existe M(z) cu $MG_1=MG_2=MG_3=MG_4=R$ .Fie $S= z_1+ z_2+z_3 +z_4$

$z_{G1}=(S-z_4)/3$ si analoagele $|z-(S-z_4)/3|=|z-(S-z_3)/3|=|z-(S-z_2)/3|=|z-(S-z_1)/3|$

Aleg $z=S / 3$ deoarece $|z_1|=|z_2|=|z_3|=|z_4|$ deci G_1, G_2, G_3, G_4 Є C(M, R) q e d.

8) Pe laturile patrulaterului convex ABCD se considera punctele M Є (AB) , N Є (BC) , P Є (CD) , Q Є (AD) astfel incat M,N,P,Q sunt mijloacele laturilor [AB] , [BC] ,[CD] ,[AD]. Aratati ca patrulaterul MNPQ este paralelogram.

Solutie: Fie $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$, $M((z_1+z_2)/2)$, $N((z_2+z_3)/2)$, $P((z_3+z_4)/2)$ $Q((z_1+z_4)/2)$. Arat ca $z_M+z_P= z_Q+z_N$, $(z_1+z_2)/2 +(z_3+z_4)/2 = (z_1+z_4)/2+(z_2+z_3)/2$.

Deci **patrulaterul MNPQ este paralelogram.**

9) Fie un hexagon regulat ABCDEF si M,N,P,Q,R ,S mijloacele laturilor AB, BC,CD, DE, EF ,FA. Sa se arate ca $NR^2=MQ^2+PS^2$ daca si numai daca $MQ \perp PS$.

Solutie: $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$, $E(z_5)$, $F(z_6)$, $NR^2=MQ^2+PS^2 \Leftrightarrow$

$|(z_5+z_6)/2 -(z_2+z_3)/2|^2=|(z_1+z_2)/2 -(z_4+z_5)/2|^2+|(z_3+z_4)/2 -(z_6+z_1)/2|^2$

$MQ \perp PS \Leftrightarrow (z_1+z_2-z_4-z_5)/(z_3+z_4-z_6-z_1) \in iR$.

Fie $a= z_1+z_2-z_4-z_5$, $b=z_3+z_4-z_6-z_1$, $z=a/b$, $z \in iR \Leftrightarrow z=-\bar{z}$, $\Leftrightarrow a/b=-\overline{a/b}$

$a\bar{b}+\bar{a}b=0$. Fie $c= z_5+z_6 - z_2-z_3$, se observa ca $a+b=-c$, $|c|^2=|a+b|^2=|a|^2+|b|^2+$ $+a\bar{b}+\bar{a}b$, deci $|c|^2=|a|^2+|b|^2$ qed.

10) Fie ΔABC si un punct M pe arcul circumscris .Notam cu H_a, H_b, H_c ortocentrele triunghiurilor BCM , ACM , ABM , notam cu E_a, E_b, E_c centrele cercurilor lui Euler si G_a, G_b, G_c centrele de greutate. Sa se arate ca :

a) $\Delta H_aH_bH_c \equiv \Delta ABC$ b) $\Delta E_aE_bE_c \approx \Delta ABC$ c) $\Delta G_aG_bG_c \approx \Delta ABC$.

Solutie: $A(z_1)$, $B(z_2)$, $C(z_3)$, $M(z)$ atunci Ω centrul cercului lui Euler are afixul $Z_\Omega=(z_1+z_2+z_3)/2$. Voi rezolva punctul b) , celelalte se fac analog

$E_a((z+z_2+z_3)/2)$, $E_b((z+z_1+z_3)/2)$, $E_c((z+z_2+z_1)/2)$, $\Delta E_aE_bE_c \approx \Delta ABC$

$\Leftrightarrow (Z_{Eb} - Z_{Ea}) / (Z_{Ec} - Z_{Ea})=(Z_1- Z_2) / (Z_1- Z_3)$ se trece la modul si se obtine $E_aE_b / AB = E_aE_c / AC$ analog se calculeaza celelalte rapoarte , deci $\Delta E_aE_bE_c \approx \Delta ABC$.

11) Consideram ΔABC inscris in cercul C(O,R) , A', B', C' punctele diametral opuse varfurilor A,B,C iar M, N,P mijloacele segmentelor $[A'B]$, $[B'C]$, $[C'A]$. Aratati ca O este centrul de greutate al ΔMNP.

Solutie:

Considerăm un reper de centru *O*. Avem: *A* (z_1), *B*(z_2), *C*(z_3), *A'*$(-z_1)$, *B'*$(-z_2)$, *C'*$(-z_3)$, .
$M((z_2-z_1)/2)$, $N((z_3-z_2)/2)$, $P((z_1-z_3)/2)$.

Centrul de greutate al triunghiului *MNP* are afixul $(z_M+z_N+z_P)/ 3= 0$.

12) **.** Fie *O* centrul pătratului *ABCD, M* mijlocul lui *BO, N* mijlocul lui *CD*. Demonstraţi că triunghiul *AMN* este dreptunghic isoscel.
Solutie:
Fie *D* originea sistemului de axe *DC* şi *DA*. Avem: *D*(0), *C*(1), *B*(1 + *i*), *A*(*i*), N(1/2), M((3(1+i)/4). Calculez Z_N-Z_M=(-3i-1)/4 si Z_A-Z_M =(i-3)/2 . Se obtine ca
Z_N-Z_M =i(Z_A-Z_M) , deci (Z_N-Z_M)/ (Z_A-Z_M) Є iR , deci AM ⊥ MN , trecem la modul si obtinem ca MN =AM. Qed.

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4. Generalizarea unei probleme date la Olimpiada Nationala de Matematica , Etapa Judeteana 2013 clasa a 6-a

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La Olimpiada judeteana de matematica 2013 clasa a 6-a a fost propusa urmatoarea problema :
Problema 3 : a) Aratati ca 20007 este un numar compus
b)Aratati ca sirul : 37, 307, 3007,…., $\underbrace{300....07}_{n}$ …. Contine o infinitate de termini care sunt numere compuse. (Numarul 300…07 are n zerouri).
Propun urmatoarea problema (o generalizare a acestei probleme) :

Problema 1

Aflati toate numerelele $\overline{pq}$ prime si numerele naturale n cu proprietatea ca :
$\overline{p00...0q}$ ⋮ $\overline{pq}$, unde numarul $\overline{p00...0q}$ are n cifre , n≥3 .
Solutie:
$\overline{p00...0q}$ =p10^{n-1}+q= p10^{n-1}-10p+10p+q=10p(10^{n-2}-1)+ $\overline{pq}$ ⋮ $\overline{pq}$, 10p(10^{n-2}-1)⋮ $\overline{pq}$, dar 10p nu e divizibil cu $\overline{pq}$ deci (10^{n-2}-1) ⋮ $\overline{pq}$. qЄ{1,3,7,9}.
In cele ce urmeaza voi folosi si teorema lui Fermat : $a^{p-1}\equiv$ 1 (mod p) , unde p este numar prim si (a, p)=1. Mai folosesc si rezultatul : “ Intr-un grup finit ,ordinul unui element x , n=ord x si x^k=e atunci n| k ”.
Cazul 1- Daca q=1 , (10^{n-2}-1) ⋮ $\overline{p1}$, pЄ{1,3,4,6,7,9}.
-Daca p=1 , (10^{n-2}-1) ⋮ 11 , $10^{10}\equiv$ 1 (mod 11) , ord(10) | 10. Dar 10^2 ≡ 1 (mod 11).
Deci (n-2) ⋮ 2 , n=2k+2 , ∀ k Є N^*.

-Daca p=3 , $(10^{n-2}-1) \vdots\ 31$, $10^{30} \equiv 1 \pmod{31}$, ord(10) | 30. Calculam $10^2 \equiv 7 \pmod{31}$, $10^3 \equiv 8 \pmod{31}$, $10^5 \equiv 25 \pmod{31}$, $10^6 \equiv 2 \pmod{31}$, $10^{10} \equiv 5 \pmod{31}$, $10^{15} \equiv 1 \pmod{31}$. Deci $(n-2) \vdots\ 15$, n=15k+2 , $\forall$ k Є N^*.
-Daca p=4 , $(10^{n-2}-1) \vdots\ 41$, $10^{40} \equiv 1 \pmod{41}$, ord(10) | 40. Calculam $10^2 \equiv 18 \pmod{41}$, $10^3 \equiv 16 \pmod{41}$, $10^4 \equiv 37 \pmod{41}$, $10^5 \equiv 1 \pmod{41}$. Deci $(n-2) \vdots\ 5$, n=5k+2 , $\forall$ k Є N^*.
-Daca p=6 , $(10^{n-2}-1) \vdots\ 61$, $10^{60} \equiv 1 \pmod{61}$, ord(10) | 60. Calculam $10^2 \equiv 39 \pmod{61}$, $10^3 \equiv 24 \pmod{61}$, $10^4 \equiv 57 \pmod{61}$, $10^5 \equiv 21 \pmod{61}$, $10^6 \equiv 27 \pmod{61}$, $10^{10} \equiv 14 \pmod{61}$, $10^{12} \equiv 58 \pmod{61}$, $10^{15} \equiv 50 \pmod{61}$, $10^{20} \equiv 13 \pmod{61}$, $10^{30} \equiv 60 \pmod{61}$, Deci $(n-2) \vdots\ 60$, n=60k+2 , $\forall$ k Є N^*.
-Daca p=7 , $(10^{n-2}-1) \vdots\ 71$, $10^{70} \equiv 1 \pmod{71}$, ord(10) | 70. Calculam $10^2 \equiv 29 \pmod{71}$, $10^3 \equiv 6 \pmod{71}$, $10^5 \equiv 32 \pmod{71}$, $10^7 \equiv 5 \pmod{71}$, $10^{10} \equiv 30 \pmod{71}$, $10^{14} \equiv 25 \pmod{71}$, $10^{35} \equiv 1 \pmod{71}$. Deci $(n-2) \vdots\ 35$, n=35k+2 , $\forall$ k Є N^*.
-Daca p=9 , $(10^{n-2}-1) \vdots\ 91$, $10^{90} \equiv 1 \pmod{91}$, ord(10) | 90. Calculam $10^2 \equiv 9 \pmod{91}$, $10^3 \equiv 90 \pmod{91}$, $10^4 \equiv 81 \pmod{91}$, $10^5 \equiv 82 \pmod{91}$, $10^6 \equiv 1 \pmod{91}$. Deci $(n-2) \vdots\ 6$, n=6k+2 , $\forall$ k Є N^*.

Cazul 2 - Daca q=3 , $(10^{n-2}-1) \vdots\ \overline{p3}$, pЄ{1,2,4,5,7,8}.

-Daca p=1 , $(10^{n-2}-1) \vdots\ 13$, $10^{12} \equiv 1 \pmod{13}$, ord(10) | 12. Calculam $10^2 \equiv 9 \pmod{13}$, $10^3 \equiv 12 \pmod{13}$, $10^4 \equiv 3 \pmod{13}$, $10^6 \equiv 1 \pmod{13}$. Deci $(n-2) \vdots\ 6$, n=6k+2 , $\forall$ k Є N^*.
-Daca p=2 , $(10^{n-2}-1) \vdots\ 23$, $10^{22} \equiv 1 \pmod{23}$, ord(10) | 22. Calculam $10^2 \equiv 8 \pmod{23}$, $10^4 \equiv 18 \pmod{23}$, $10^6 \equiv 6 \pmod{23}$, $10^{10} \equiv 16 \pmod{23}$, $10^{11} \equiv 22 \pmod{23}$. Deci $(n-2) \vdots\ 22$, n=22k+2 , $\forall$ k Є N^*.
-Daca p=4 , $(10^{n-2}-1) \vdots\ 43$, $10^{42} \equiv 1 \pmod{43}$, ord(10) | 42. Calculam $10^2 \equiv 14 \pmod{43}$, $10^3 \equiv 11 \pmod{43}$, $10^6 \equiv 35 \pmod{43}$, $10^7 \equiv 6 \pmod{43}$, $10^{14} \equiv 36 \pmod{43}$, $10^{21} \equiv 1 \pmod{43}$. Deci $(n-2) \vdots\ 21$, n=21k+2 , $\forall$ k Є N^*.
- Daca p=5 , $(10^{n-2}-1) \vdots\ 53$, $10^{52} \equiv 1 \pmod{53}$, ord(10) | 52. Calculam $10^2 \equiv 47 \pmod{53}$, $10^4 \equiv 36 \pmod{53}$, $10^8 \equiv 24 \pmod{53}$, $10^{12} \equiv 16 \pmod{53}$, $10^{13} \equiv 1 \pmod{53}$. Deci $(n-2) \vdots\ 13$, n=13k+2 , $\forall$ k Є N^*.
- Daca p=7 , $(10^{n-2}-1) \vdots\ 73$, $10^{72} \equiv 1 \pmod{73}$, ord(10) | 72. Calculam $10^2 \equiv 27 \pmod{73}$, $10^3 \equiv 51 \pmod{73}$, $10^4 \equiv 72 \pmod{73}$, $10^6 \equiv 46 \pmod{73}$, $10^8 \equiv 1 \pmod{73}$. Deci $(n-2) \vdots\ 8$, n=8k+2 , $\forall$ k Є N^*.
- Daca p=8 , $(10^{n-2}-1) \vdots\ 83$, $10^{82} \equiv 1 \pmod{83}$, ord(10) | 82. Calculam $10^2 \equiv 17 \pmod{83}$, $10^3 \equiv 4 \pmod{83}$, $10^6 \equiv 16 \pmod{83}$, $10^{12} \equiv 7 \pmod{83}$, $10^{28} \equiv 51 \pmod{83}$, $10^{40} \equiv 25 \pmod{83}$, $10^{41} \equiv 1 \pmod{83}$.

Cazul 3 - Daca q=7 , $(10^{n-2}-1) \vdots\ \overline{p7}$, pЄ{1,3,4,6,9}.

- Daca p=1 , $(10^{n-2}-1) \vdots\ 17$, $10^{16} \equiv 1 \pmod{17}$, ord(10) | 16. Calculam $10^2 \equiv 15 \pmod{17}$, $10^4 \equiv 4 \pmod{17}$, $10^8 \equiv 16 \pmod{17}$. Deci $(n-2) \vdots\ 16$, n=16k+2 , $\forall$ k Є N^*.
- Daca p=3 , $(10^{n-2}-1) \vdots\ 37$, $10^{36} \equiv 1 \pmod{37}$, ord(10) | 36. Calculam $10^2 \equiv 26 \pmod{37}$, $10^3 \equiv 1 \pmod{37}$. Deci $(n-2) \vdots\ 3$, n=3k+2 , $\forall$ k Є N^*.
- Daca p=4 , $(10^{n-2}-1) \vdots\ 47$, $10^{46} \equiv 1 \pmod{47}$, ord(10) | 46. Calculam $10^2 \equiv 53 \pmod{47}$, $10^4 \equiv 36 \pmod{47}$, $10^8 \equiv 27 \pmod{47}$, $10^{16} \equiv 24 \pmod{47}$, $10^{20} \equiv 18 \pmod{47}$, $10^{23} \equiv 46 \pmod{47}$. Deci $(n-2) \vdots\ 46$, n=46k+2 , $\forall$ k Є N^*.

- Daca p=6, $(10^{n-2}-1) \vdots\ 67$, $10^{66} \equiv 1 \pmod{67}$, ord(10) | 66. Calculam $10^2 \equiv 33 \pmod{67}$, $10^3 \equiv 62 \pmod{67}$, $10^6 \equiv 25 \pmod{67}$, $10^{11} \equiv 29 \pmod{67}$, $10^{22} \equiv 37 \pmod{67}$, $10^{33} \equiv 1 \pmod{67}$. Deci $(n-2) \vdots\ 33$, $n=33k+2$, $\forall\ k \in N^*$.
- Daca p=9 , $(10^{n-2}-1) \vdots\ 97$, $10^{96} \equiv 1 \pmod{97}$, ord(10) | 96. Calculam $10^2 \equiv 3 \pmod{97}$, $10^3 \equiv 30 \pmod{97}$, $10^4 \equiv 9 \pmod{97}$, $10^6 \equiv 27 \pmod{97}$, $10^{12} \equiv 50 \pmod{97}$, $10^{24} \equiv 75 \pmod{97}$, $10^{48} \equiv 96 \pmod{97}$. Deci $(n-2) \vdots\ 96$, $n=96k+2$, $\forall\ k \in N^*$.

Cazul 4 - Daca q=9 , $(10^{n-2}-1) \vdots\ \overline{p9}$, p∈{1,2,5,7,8}.

- Daca p=1 , $(10^{n-2}-1) \vdots\ 19$, $10^{18} \equiv 1 \pmod{19}$, ord(10) | 18. Calculam $10^2 \equiv 5 \pmod{19}$, $10^3 \equiv 12 \pmod{19}$, $10^6 \equiv 11 \pmod{19}$, $10^9 \equiv 18 \pmod{19}$.Deci $(n-2) \vdots\ 18$, $n=18k+2$, $\forall\ k \in N^*$.
-Daca p=2 , $(10^{n-2}-1) \vdots\ 29$, $10^{28} \equiv 1 \pmod{29}$, ord(10) | 28. Calculam $10^2 \equiv 13 \pmod{29}$, $10^4 \equiv 24 \pmod{29}$, $10^7 \equiv 17 \pmod{29}$, $10^{14} \equiv 28 \pmod{29}$. Deci $(n-2) \vdots\ 28$, $n=28k+2$, $\forall\ k \in N^*$.
- Daca p=5 , $(10^{n-2}-1) \vdots\ 59$, $10^{58} \equiv 1 \pmod{59}$, ord(10) | 58. Calculam $10^2 \equiv 41 \pmod{59}$, $10^4 \equiv 29 \pmod{59}$, $10^8 \equiv 15 \pmod{59}$, $10^{16} \equiv 48 \pmod{59}$, $10^{20} \equiv 35 \pmod{59}$, $10^{29} \equiv 58 \pmod{59}$. Deci $(n-2) \vdots\ 58$, $n=58k+2$, $\forall\ k \in N^*$.
-Daca p=7 , $(10^{n-2}-1) \vdots\ 79$, $10^{78} \equiv 1 \pmod{79}$, ord(10) | 78. Calculam $10^2 \equiv 21 \pmod{79}$, $10^3 \equiv 52 \pmod{79}$, $10^6 \equiv 18 \pmod{79}$, $10^{13} \equiv 1 \pmod{79}$. Deci $(n-2) \vdots\ 13$, $n=13k+2$, $\forall\ k \in N^*$.
- Daca p=8 , $(10^{n-2}-1) \vdots\ 89$, $10^{88} \equiv 1 \pmod{89}$, ord(10) | 88. Calculam $10^2 \equiv 11 \pmod{89}$, $10^4 \equiv 32 \pmod{89}$, $10^8 \equiv 45 \pmod{89}$, $10^{11} \equiv 55 \pmod{89}$, $10^{22} \equiv 88 \pmod{89}$, $10^{44} \equiv 1 \pmod{89}$. Deci $(n-2) \vdots\ 44$, $n=44k+2$, $\forall\ k \in N^*$.

Problema 2

a) Aratati ca numarul $\overline{p\underbrace{00...0}_{n}q}$ este compus , pentru orice numar prim $\overline{pq}$, pentru o infinitate de valori ale lui n. (unde n≥3).

b) b)Aratati ca sirul : $\overline{pq}$, $\overline{p0q}$,......., $\overline{p\underbrace{00...0}_{n}q}$ contine o infinitate de numere compuse , pentru orice numar prim $\overline{pq}$, pentru o infinitate de valori ale lui n. (unde n≥3).

Solutie:

Arat ca numarul $\overline{p00...0q} \vdots\ \overline{pq}$, $\forall\ \overline{pq}$ numar prim si caut pe n .

$\overline{p00...0q} = p10^{n-1}+q= p10^{n-1}-10p+10p+q=10p(10^{n-2}-1)+ \overline{pq} \vdots\ \overline{pq}$, $10p(10^{n-2}-1) \vdots\ \overline{pq}$, dar 10p nu e divizibil cu $\overline{pq}$ deci $(10^{n-2}-1) \vdots\ \overline{pq}$.

In cele ce urmeaza voi folosi si teorema lui Fermat : $a^{p-1} \equiv 1 \pmod{p}$, unde p este numar prim si (a, p)=1. Aplic Teorema lui Fermat :

($\overline{pq}$ - 1) ,

Deci $(n-2) \vdots\ (\overline{pq}-1)$, $n=(\overline{pq}-1)k+2$, $\forall\ k \in N^*$.

$10 \equiv 1 \pmod{\overline{pq}}$.

Am gasit o infinitate de valori ale lui n.

Bibliografie: Subiect dat la Olimpiada Judeteana de Matematica , clasa a 6-a , 2013.