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Coordonator: Andrei Octavian Dobre
E-mail pentru articole:
revistaelectronica@mateinfo.ro

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1. Other solutions for the problem 202 and the problem 204 from La Gaceta de la RSME

by Roxana – Mihaela Stanciu[2] and Nela Ciceu[3]

Abstract. *The solutions of the problem 202 and the problem 204 was presented in La Gaceta de la RSME, Vol. 16 (2013), No.2, pp. 289-292. Here we present new solutions for this two problems.*

PROBLEMA 202. *Propuesto por Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italia.*

Para un triángulo ABC denotaremos por r su inradio, por r_a, r_b y r_c sus exinradios, por I su incentro, y por I_a, I_b e I_c sus exincentros. Probar o refutar la desigualdad

$$\frac{\cos A}{1-\cos^2 A}+\frac{\cos B}{1-\cos^2 B}+\frac{\cos C}{1-\cos^2 C}\geq \frac{1}{4r}\sqrt{\frac{II_a\cdot II_b\cdot II_c}{r_ar_br_c}(r_ar_b+r_br_c+r_cr_a)}.$$

Solution:

Since we well-known that:

$$II_a=4R\sin\frac{A}{2}\ ,\ \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\text{ and }r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

We deduce that:

$$\frac{II_a\cdot II_b\cdot II_c}{r_ar_br_c}(r_ar_b+r_br_c+r_cr_a)=\frac{64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{r}=16R^2,$$

And we have to prove that:

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq\frac{R}{r}\Leftrightarrow(1)\frac{\cos A}{4R^2\sin^2 A}+\frac{\cos B}{4R^2\sin^2 B}+\frac{\cos C}{4R^2\sin^2 C}\geq\frac{1}{4Rr}$$

$$\Leftrightarrow\frac{\cos A}{a^2}+\frac{\cos B}{b^2}+\frac{\cos C}{c^2}\geq\frac{s}{abc}\Leftrightarrow(2)\frac{b^2+c^2-a^2}{2a^2bc}+\frac{c^2+a^2-b^2}{2ab^2c}+\frac{a^2+b^2-c^2}{2abc^2}\geq\frac{s}{abc}$$

$$\Leftrightarrow\frac{b^2+c^2}{a}-a+\frac{c^2+a^2}{b}-b+\frac{a^2+b^2}{c}-c\geq 2s\Leftrightarrow\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c}\geq 4s\quad(*)$$

We deduce (1) by the law of sines and (2) by the law of cosines.

By *Bergström'* s inequality we deduce that:

$$\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}\geq\frac{(a+b+c)^2}{a+b+c}=2s\text{ and }\frac{c^2}{a}+\frac{a^2}{b}+\frac{b^2}{c}\geq\frac{(a+b+c)^2}{a+b+c}=2s,$$

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[2] Liceul cu Program Sportiv ''Iolanda Balas Soter''

[3] Roșiori, Bacău

which by adding yields to (*) and we are done.
We have equality if and only if $a = b = c$.

PROBLEMA 204. *Propuesto por Juan Bosco Romero Márquez, Universidad Complutense de Madrid, Madrid.*

Sea ABC un triángulo y, con las notaciones usuales, definimos la cantidad

$$d = rr_a + r_b r_c - 2m_a h_a.$$

Establecer condiciones suficientes sobre los ángulos del triángulo ABC para que la cantidad d sea, respectivamente, positiva, negativa o nula.

Solution :

We denote by $S = area(ABC)$ and $p = semiperimeter(ABC)$, and by well-known formuls we obtain that:

$$d = \frac{S^2}{p(p-a)} + \frac{S^2}{(p-b)(p-c)} - 2m_a h_a = \frac{S^2(p^2 - pb - pc + bc + p^2 - pa)}{p(p-a)(p-b)(p-c)} - 2m_a h_a =$$

$$= 2p^2 - p(a+b+c) + bc - 2m_a h_a = bc - 2m_a h_a = \frac{2S}{\sin A} - 2m_a \cdot \frac{2S}{a} = \frac{2S}{a}\left(\frac{a}{\sin A} - 2m_a\right),$$

i.e.

$$d = \frac{4S}{a}(R - m_a).$$

To compare d with 0 returns to study the sign of the expresion $R - m_a$ according to the angles of the triangle ABC, and this is the subject of the problem 3113 from CRUX MATHEMATICORUM, with solution in no. 1/2007, pp. 63-64 (see the solution from the pages below).We are done.

3113. [2006 : 47, 49; 171, 174] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

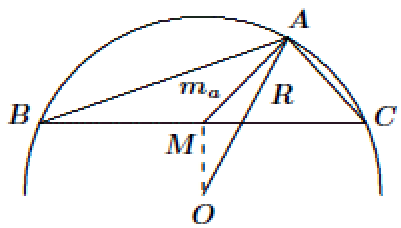
Let ABC be a triangle and let a be the length of the side opposite the vertex A. If m_a is the length of the median from A to BC, and if R is the circumradius of $\triangle ABC$, prove that $m_a - R$ is positive, negative, or zero, according as $\angle A$ is obtuse, acute, or right-angled.

Combination of similar solutions by Roy Barbara, University of Beirut, Beirut, Lebanon; and Richard I. Hess, Rancho Palos Verdes, CA, USA.

We let M be the mid-point of BC and consider the triangle OMA with sides $AM = m_a$ and $AO = R$. According to Euclid, the relative sizes of these two sides depends on the size of the opposite angles.

Case 1. A is obtuse.

Vertex A (on the circumcircle) is separated from the circumcentre O by the chord BC. Since $OM \perp BC$, $\angle OMA$ is obtuse; whence, the opposite side R is longer than the adjacent side m_a; that is, $m_a - R < 0$ when A is obtuse, as claimed.

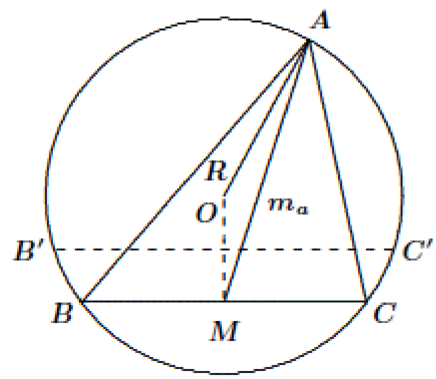


Case 2. $A = 90°$.

Here BC is a diameter; thus, $m_a = R$, and $m_a - R = 0$.

Case 3. A is acute.

The proposal is incorrect: $m_a - R$ can be positive, zero, or negative when A is acute, as follows. Let $B'C'$ be the perpendicular bisector of OM. For A on the long arc of the circumcircle between B' and C', we have $\angle MOA > \angle OMA$; whence $m_a - R > 0$. For A at B' or at C', we get $m_a - R = 0$. Finally, when A lies on either short arc $B'B$ or $C'C$, we see that $m_a - R < 0$. Note that the proposal becomes correct for triangles ABC in which *all* angles are acute; then A will necessarily lie on the arc $B'C'$ which, as we have just seen, forces $m_a - R > 0$, as claimed in the proposal.



*Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; *VEDULA N. MURTY, Dover, PA, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania (with two proofs for the obtuse-angle case); and the *proposer. The asterisk designates solutions that were correct, but whose analysis of the acute-angle case was incomplete. In addition VÁCLAV KONEČNÝ, Big Rapids, MI, USA provided a counterexample showing that the conclusion to the corrected proposal still was flawed. There were three incorrect submissions.*

2. THE SOLUTIONS OF SOME PROBLEMS OF MATHEMATICAL REFLECTIONS

IOAN VIOREL CODREANU, Secondary School Satulung, Maramures

J 253 Prove that if $a,b,c>0$ satisfy $abc=1$, then

$$\frac{1}{ab+a+2}+\frac{1}{bc+b+2}+\frac{1}{ca+c+2}\le\frac{3}{4}$$

Proposed by Marcel Chiriță, Bucharest, Romania

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

With the substitutions $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}, x,y,z>0$, the inequality is written

$$\sum\frac{zy}{xy+xz+2zy}\le\frac{3}{4}.$$

Let $s=xy+yz+zx$ and $s+zy=\alpha, s+xz=\beta, s+xy=\gamma$. We have

$$\sum\frac{zy}{xy+xz+2zy}=\sum\frac{zy}{s+zy}=\sum\frac{\alpha-s}{\alpha}=3-s\sum\frac{1}{\alpha}\le 3-s\cdot\frac{9}{\sum\alpha}=3-s\cdot\frac{9}{4s}=\frac{3}{4},$$

where we used the inequality $\left(\sum\alpha\right)\left(\sum\frac{1}{\alpha}\right)\ge 9$.

J 256 Evaluate

$$1^2\cdot 2!+2^2\cdot 3!+\ldots+n^2(n+1)!$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

We have

$$\begin{aligned}k^2(k+1)!&=\left(k^2+4k+4-4k-4\right)(k+1)!=\\&=(k+2)^2(k+1)!-4(k+1)(k+1)!=\\&=(k+2)(k+2)!-4(k+1)(k+1)!,\ \forall k=\overline{1,n}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=1}^{n}k^2(k+1)!&=\sum_{k=1}^{n}\left[(k+2)(k+2)!-4(k+1)(k+1)!\right]=\\&=\sum_{k=1}^{n}\left[(k+3)-1\right](k+2)!-4\sum_{k=1}^{n}\left[(k+2)-1\right](k+1)!=\\&=\sum_{k=1}^{n}\left[(k+3)!-(k+2)!\right]-4\sum_{k=1}^{n}\left[(k+2)!-(k+1)!\right]=\\&=(n+3)!-3!-4\left[(n+2)!-2!\right]=\\&=(n-1)(n+2)!+2.\end{aligned}$$

J 260. Solve in integers the equation

$$x^4-y^3=111$$

Proposed by Jose Hernandez Santiago, Oaxaca, Mexico

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

We have $x^4\equiv 0,1,3,9(\bmod 13)$ and $y^3\equiv 0,1,5,8,12(\bmod 13)$. Then

$$x^4-y^3\equiv 0,1,2,3,4,5,6,8,9,10,11,12(\bmod 13)$$

and because $111\equiv 7(\bmod 13)$ the equation $x^4-y^3=111$ does not have solutions.

S 262. Let a,b,c be the sides of a triangle and let m_a, m_b, m_c be the lengths of its medians. Prove that

$$a^2+b^2+c^2-ab-bc-ca \le 4\left(m_a{}^2+m_b{}^2+m_c{}^2-m_am_b-m_bm_c-m_cm_a\right)$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the identity $\sum m_a{}^2=\frac{3}{4}\sum a^2$ the inequality $\sum a^2-\sum ab \le 4\left(\sum m_a{}^2-\sum m_am_b\right)$ is equivalent to $4\sum m_am_b \le 2\sum a^2+\sum ab$. For any triangle we have the relation

$$4m_bm_c \le 2a^2+bc$$

Indeed we have successively

$$4m_bm_c \le 2a^2+bc \Leftrightarrow 16m_b{}^2m_c{}^2 \le 4a^4+4a^2bc+b^2c^2 \Leftrightarrow$$
$$\left[2\left(a^2+c^2\right)-b^2\right]\left[2\left(a^2+b^2\right)-c^2\right] \le 4a^4+4a^2bc+b^2c^2$$

and after the calculations, the last one inequality is writtten

$$a^2b^2+a^2c^2+2b^2c^2 \le 2a^2bc+b^4+c^4 \Leftrightarrow a^2(b-c)^2-\left(b^2-c^2\right) \le 0 \Leftrightarrow$$
$$(b-c)^2\left(a^2-(b+c)^2\right) \le 0 \Leftrightarrow (b-c)^2(a-b-c)(a+b+c) \le 0$$

The last one inequality is true , so

$$4\sum m_am_b \le 2\sum a^2+\sum ab$$

and the solution ends.

S 264. Let a,b,c,x,y,z be positive real numbers such that

$ab+bc+ca=xy+yz+zx=1$.

Prove that

$$a(y+z)+b(z+x)+c(x+y) \ge 2$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

We wil show that $\sum a(y+z) \geq 2\sqrt{\left(\sum ab\right)\left(\sum xy\right)}$ and the inequality from the statement follows easely. The inequality is homogeneous in x, y and z, so we may assume that $\sum x = 1$. We rewrite inequality as follows

$$\sum a \geq \sum ax + 2\sqrt{\left(\sum ab\right)\left(\sum xy\right)}$$

We apply the Cauchy- Schwarz Inequality to obtain

$$\sum ax \leq \sqrt{\left(\sum a^2\right)\left(\sum x^2\right)}$$

Applying the Cauchy-Schwarz Inequality one more time we get

$$\sqrt{\left(\sum a^2\right)\left(\sum x^2\right)} + \sqrt{\left(\sum ab\right)\left(\sum xy\right)} + \sqrt{\left(\sum ab\right)\left(\sum xy\right)} \leq \sqrt{\left(\sum a^2 + 2\sum ab\right)\left(\sum x^2 + 2\sum xy\right)} =$$
$$\sqrt{\left(\sum a\right)^2\left(\sum x\right)^2} = \sum a$$

So

$$\sum a \geq \sum ax + 2\sqrt{\left(\sum ab\right)\left(\sum xy\right)}$$

and the solution ends.

J 248. ***Let*** $f:[1,\infty) \to R$ ***be defined by*** $f(x) = \frac{\{x\}^2}{[x]}$. ***Prove that*** $f(x+y) \leq f(x) + f(y)$, ***for***

any real numbers x ***and*** y.

Proposed by Sorin Rădulescu, Bucharest, Romania

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Lemma 1. ***For any*** $x \in R$ ***and any*** $k \in Z$, ***we have:***

$$k \leq x \Leftrightarrow k \leq [x].$$

Proof. From $k \leq x$ we get $k \leq x < [x]+1$ which means that $[x] > k-1$. From here we deduce, by taking into account that $[x]$ and k are integers that $[x] \geq k$. Let $x \in R$ with $[x] \geq k$. We have then $x \geq [x] \geq k$ and the equivalence is demonstrated.

Lemma 2. ***For any*** $x, y \in R$, ***we have:***

$$[x]+[y]\le[x+y] \textbf{ and } \{x\}+\{y\}\ge\{x+y\}.$$

Proof. Let $[x]=h$ and $[y]=k$. We have $x\ge h$ and $y\ge k$ whence using **Lemma 1**, we get

$[x+y]\ge h+k$. The inequality $[x]+[y]\le[x+y]$ is equivalent to

$x-\{x\}+y-\{y\}\le x+y-\{x+y\}$ whence we get $\{x\}+\{y\}\ge\{x+y\}$ which concludes the prof of Lemma.

By applying the **Bergström Inequality**, using **Lemma 2** we get

$$f(x)+f(y)=\frac{\{x\}^2}{[x]}+\frac{\{y\}^2}{[y]}\ge\frac{(\{x\}+\{y\})^2}{[x]+[y]}\ge\frac{\{x+y\}^2}{[x+y]}=f(x+y),\forall x,y\in[1,\infty)$$

and the solution is completed.

J 250. ***Let ABC be a triangle with $\angle A\ge 120^\circ$ and let s be the semiperimeter of the triangle. Prove that***

$$\sqrt{(s-b)(s-c)}\ge\left(3+\sqrt{6}\right)(s-a)$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the **Ravi** substitutions $x=s-a, y=s-b, z=s-c$ the inequality of the statement is written

$$\sqrt{yz}\ge\left(3+\sqrt{6}\right)x \quad (1).$$

From $\angle A\ge 120^\circ$ we get $\cos A\le-\frac{1}{2}$ (2). Using the **Law of Cosines** and (2) we have

$$a^2=b^2+c^2-2bc\cos A\ge b^2+c^2+bc \quad (3).$$

Substituting in (3), $a=y+z, b=z+x, c=x+y$ we get the inequality

$$(y+z)^2\ge(z+x)^2+(x+y)^2+(z+x)(x+y)$$

equivalent to

$$yz\ge 3x^2+3x(y+z) \quad (4).$$

We note $\sqrt{yz}=t$. From (4), using the **AM-GM Inequality**: $y+z\ge 2\sqrt{yz}$ we get

$$t^2-6xt-3x^2\ge 0 \quad (5).$$

The equation (with the unknown t) $t^2 - 6xt - 3x^2 = 0$ has the discriminant $\Delta = 48x^2$ and the solutions $t_1 = (3+2\sqrt{3})x > 0$ and $t_2 = (3-2\sqrt{3})x < 0$. Taking account of $t > 0$, the inequality (5) has the solution $t \geq (3+2\sqrt{3})x$ and how $(3+2\sqrt{3})x > (3+\sqrt{6})x$, we get (1) and the solution is completed.

J 251. ***Let*** a, b, c ***be positive real numbers such that*** $a \geq b \geq c$ ***and*** $b^2 > ac$**.**

Prove that

$$\frac{1}{a^2 - bc} + \frac{1}{b^2 - ac} + \frac{1}{c^2 - ab} > 0$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

From $a \geq b \geq c$ it follows that $a^2 \geq bc$ and $c^2 \leq ab$. Has no place $a^2 = bc$ or $c^2 = ab$ because every time it is obtained $a = b = c$ and contradicts the condition $b^2 > ac$. So, we have $a^2 - bc > 0, b^2 - ac > 0, c^2 - ab < 0$ and then the inequality of the statement is equivalent to

$$\frac{(c^2 - ab)(b^2 - ac) + (a^2 - bc)(c^2 - ab) + (a^2 - bc)(b^2 - ac)}{(a^2 - bc)(b^2 - ac)(c^2 - ab)} > 0$$

and with

$$(c^2 - ab)(b^2 - ac) + (a^2 - bc)(c^2 - ab) + (a^2 - bc)(b^2 - ac) < 0.$$

After opening of the parentheses, the last inequality becomes

$$a^2b^2 + b^2c^2 + c^2a^2 + a^2bc + ab^2c + abc^2 < a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3 \quad (1)$$

Using the **AM-GM Inequality** we get

$$a^3b + ab^3 \geq 2a^2b^2, b^3c + bc^3 \geq 2b^2c^2, c^3a + ca^3 \geq 2c^2a^2$$

and after adding these inequalities, we get

$$a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3 > 2a^2b^2 + 2b^2c^2 + 2c^2a^2$$

The last inequality is strict, othervise $a = b = c$ and contradicts the condition $b^2 > ac$.

To prove the inequality (1) it is sufficient to prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + ab^2c + abc^2 \quad (2).$$

Using the **Cauchy-Schwarz Inequality** we get

$$\left(a^2b^2+b^2c^2+c^2a^2\right)\left(b^2c^2+c^2a^2+a^2b^2\right)\geq\left(abbc+bcca+caab\right)^2$$

and

$$a^2b^2+b^2c^2+c^2a^2\geq a^2bc+ab^2c+abc^2$$

namely the inequality (2) is true and the solution ends.

S 247 ***Prove that for any positive integers*** m ***and*** n***, the number*** $8m^6+27m^3n^3+27n^6$ ***is***

composite.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the well known identity

$$x^3+y^3+z^3-3xyz=(x+y+z)\left(x^2+y^2+z^2\right)$$

for $x=2m^2, y=3n^2$ and $z=-3mn$, we have

$$8m^6+27m^3n^3+27n^6=\left(2m^2\right)^3+\left(3n^2\right)^3+(-3mn)^3-3\left(2m^2\right)\left(3n^2\right)(-3mn)=$$
$$=\left(2m^2+3n^2-3mn\right)\left(4m^4+9n^4+3m^2n^2+9mn^3+6m^3n\right).$$

How $2m^2+3n^2=2\left(m^2+n^2\right)+n^2\geq 4mn+n^2=3mn+\left(mn+n^2\right)\geq 3mn+2$, namely $2m^2+3n^2-3mn\geq 2$ and $4m^4+9n^4+3m^2n^2+9mn^3+6m^3n\geq 2$ it is obviously true, it follows that the number $8m^6+27m^3n^3+27n^6$ is composite.

S 261. ***Fiind all triples*** (x,y,z) ***of positive real numbers for which there is a positive real***

number t ***such that the following inequalities hold simultaneously:***

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+t\leq 4, x^2+y^2+z^2+\frac{2}{t}\leq 5.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution 1 by Ioan Viorel Codreanu, Satulung, Maramures, Romania

From $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+t\leq 4$ we get $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+2t\leq 8$ and how $x^2+y^2+z^2+\frac{2}{t}\leq 5$ it follows that

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+2t+x^2+y^2+z^2+\frac{2}{t}\le 13 \quad (1).$$

Using the **AM-GM Inequality** and the inequality (1), we get

$$13\ge\frac{1}{x}+\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+\frac{1}{z}+\frac{1}{z}+t+t+x^2+y^2+z^2+\frac{1}{t}+\frac{1}{t}\ge 13\sqrt[13]{\frac{1}{(xyz)^2}\cdot t^2\cdot (xyz)^2\cdot\frac{1}{t^2}}=13.$$

Therefore, we have equality in the **AM-GM Inequality**. Then

$$\frac{1}{x}=\frac{1}{y}=\frac{1}{z}=x^2=y^2=z^2=t=\frac{1}{t}$$

whence we get $x=y=z=t=1$, so $(x,y,z)=(1,1,1)$.

Solution 2 by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the **AM-GM Inequality** we get

$$4\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+t\ge 4\sqrt[4]{\frac{t}{xyz}}$$

whence it follows that $\frac{t}{xyz}\le 1$ (1).

Using the **AM-GM Inequality** we get

$$5\ge x^2+y^2+z^2+\frac{1}{t}+\frac{1}{t}\ge 5\sqrt[5]{\frac{x^2y^2z^2}{t^2}}$$

whence it follows that $\frac{t}{xyz}\ge 1$ (2).

From (1) and (2) we get $t=xyz$ (3). Using the inequalities of enunciation, the **AM-GM Inequality** and the equality (3), we get

$$5-t\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\ge\frac{4}{\sqrt[4]{xyz}}=\frac{4}{\sqrt[4]{t}} \quad (4)$$

and

$$5-\frac{1}{t}\ge x^2+y^2+z^2+\frac{1}{t}\ge 4\sqrt[4]{\frac{x^2y^2z^2}{t}}=4\sqrt[4]{t} \quad (5).$$

From (4) and (5), taking account of the inequalities $t+\frac{1}{t}\ge 2$ and $\sqrt[4]{t}+\frac{1}{\sqrt[4]{t}}\ge 2$ we get

$$8 \ge 10 - \left(t + \frac{1}{t}\right) \ge 4\left(\sqrt[4]{t} + \frac{1}{\sqrt[4]{t}}\right) \ge 8$$

so, we have equality in the previous inequality, namely $t = \frac{1}{t}$ and then $t = xyz = 1$ (6).

Substituting $t = 1$ in the inequality of enunciation, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le 3 \quad (7).$$

Using the **AM-GM Inequality**, the inequality (7) and equalities (6), we have

$$3 \ge \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{3}{\sqrt[3]{xyz}} = 3$$

so, we have equality in the previous inequality, namely $x = y = z$ and then $(x, y, z) = (1,1,1)$.

O 249. ***Find all triples*** (x, y, z) ***of positive integers such that***

$$\frac{x}{y} + \frac{y}{z+1} + \frac{z}{x} = \frac{5}{2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Let (x, y, z) a solution of the given equation. Using the **AM-GM Inequality** we get

$$\frac{5}{2} = \frac{x}{y} + \frac{y}{z+1} + \frac{z}{x} \ge 3\sqrt[3]{\frac{x}{y} \cdot \frac{y}{z+1} \cdot \frac{z}{x}} = 3\sqrt[3]{\frac{z}{z+1}}$$

whence it follows that

$$\frac{z}{z+1} \le \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

and by solving the inequality we get $z \le \frac{125}{91}$, namely $z = 1$.

Substituting $z = 1$, we write the given equation in the form

$$\frac{y}{2} = \frac{5}{2} - \frac{x}{y} - \frac{1}{x} < \frac{5}{2}$$

and we deduce that $y \le 4$.

For $y=1$ the equation becomes $x+\frac{1}{x}=2$ with the solution $x=1$. For $y=2$ the equation becomes $\frac{x}{2}+\frac{1}{x}=\frac{3}{2}$ equivalent to $x^2-3x+2=0$ with the solutions $x=1$ and $x=2$. For $y=3$ the equation becomes $\frac{x}{3}+\frac{1}{x}=1$ equivalent to $x^2-3x+3=0$ without solutions in integers. For $y=4$ the equation becomes $\frac{x}{4}+\frac{1}{x}=\frac{1}{2}$ equivalent to $x^2-2x+4=0$ without solutions in integers. Triples $(1,1,1),(1,2,1)$ and $(2,2,1)$ verifies the given equation, so

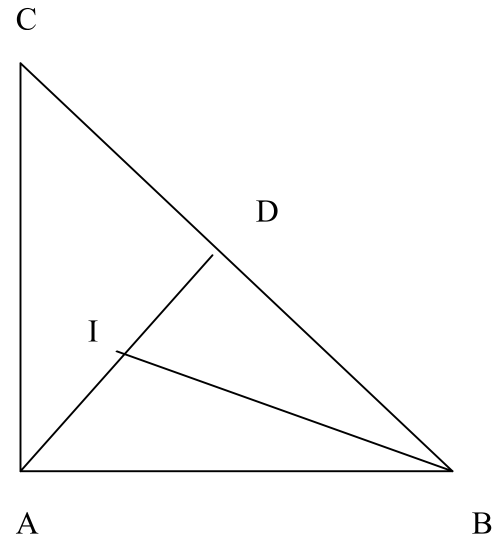
$(x,y,z)\in\{(1,1,1),(1,2,1),(2,2,1)\}$.

3. Asupra unei probleme date la Olimpiada Nationala de Matematica 2013- clasa a 6-a

Profesor Serban George-Florin
Liceul Tehnologic "Grigore Moisil " Braila

-La olimpiada nationala de matematica 2013 clasa a 6-a a fost propusa urmatoarea problema de geometrie :
"Se considera triunghiul ABC cu AB=AC si m($\angle BAC$)=90^0 . Fie D€ (BC) astfel incat AD $\perp$ BC .Bisectoarea unghiului ABC intersecteaza dreapta AD in punctul I . Demonstrati ca AI +AB=BC" .
Voi prezenta in continuare 10 metode de rezolvare a acestei probleme si cele 3 reciproce.

Metoda 1 : (sintetica)
ΔABC isoscel AD inaltime deci AD este bisectoare . Rezulta ca I este centrul cercului inscris in ΔABC .
Fie r =raza cercului inscris in ΔABC r =DI
Calculez r = S / p ,
AB=AC=a , BC=a√2 (cu T. Pitagora). S= a^2 / 2 iar p=(2a +a√2) /2. Se obtine ca r = a(2-√2) / 2. Dar AI=AD – DI , AD este mediana in triunghi dreptunghic deci



AD=BC/2= a√2 /2 se obtine AI=a√2 –a =BC – AB qed .

Metoda 2 : (sintetica)

In ΔABD aplic teorema bisectoarei $\frac{AI}{DI}=\frac{AB}{BD}$, $\frac{AI}{DI}=\sqrt{2}$, $\frac{AI}{DA}=\sqrt{2}/(\sqrt{2}+1)$

AD este mediana in triunghi dreptunghic deci AD=BC/2= a√2 /2
Gasim ca AI= a√2 –a =BC – AB qed .

Metoda 3 : (analitica)

In reperul cartezian XOY consider punctele A=O (0,0) , B(a ,0) , C(0,a) , D(a/2 , a/2).
a >0. ΔABC isoscel AD inaltime deci AD este bisectoare , mediana .
Ecuatia dreptei AD este prima bisectoare y=x . Calculez tg $22^0 30^{|}$ folosind formula Tg(X / 2)= (sinX) / (1+cos X) , pentru X=45^0. Fie m=panta dreptei BI ,se gaseste ca m=tg (180^0-$22^0$30)= - tg $22^0 30^{|}$, m=1 -√2 .
Ecuatia dreptei BI : $y-y_0=m(x-x_0)$, y=(x-a)(1 -√2). Dar {I}=AD∩ BI , rezolvand sistemul de ecuatii format cu ecuatiile celor doua drepte vom obtine ca I (a - a√2 /2 , a - a√2 /2) . Aplcand formula distantei dintre doua puncte in plan gasim ca AI= a√2 –a =BC – AB qed .

Metoda 4 : (analitica)

In reperul cartezian XOY consider punctele A=O (0,0) , B(a ,0) , C(0,a) , a >0.
D(a/2 , a/2). In ΔABD aplic teorema bisectoarei , gasim ca $\frac{AI}{DI}=\sqrt{2}$.
I($(x_1+kx_2)/(1+k)$, $(y_1+ky_2)/(1+k)$) , unde k=√2 gasim ca I (a - a√2 /2 , a - a√2 /2)
Aplcand formula distantei dintre doua puncte in plan gasim ca AI= a√2 –a =BC – AB qed

Metoda 5 : (cu afixe)

Fie $A(z_1)$, $B(z_2)$, $C(z_3)$, $\frac{AI}{DI}=\sqrt{2}$, $D((z_2+z_3)/2)$, $I([z_1+\sqrt{2}(z_2+z_3)/2]/(1+\sqrt{2}))$

Se gaseste ca I ($[(2\sqrt{2}-2)z_1+(2-\sqrt{2})(z_2+z_3)]/2$)
AI=$|z_1-[(2\sqrt{2}-2)z_1+(2-\sqrt{2})(z_2+z_3)]/2|$ .Se obtine ca AI=$(2-\sqrt{2})|2z_1-z_2-z_3|/2$)
Arat ca $|2z_1-z_2-z_3|=a\sqrt{2}=|z_2-z_3|$, a=AB=AC=$|z_2-z_1|=|z_1-z_3|$
Notez cu $u=z_1-z_3$ si $v=z_1-z_2$, $|u+v|^2=|u|^2+|v|^2+\bar{u}v+u\bar{v}=2a^2$,

Obtinem ca $\bar{u}v + u\bar{v} = 0$, $\bar{u} / \bar{v} = -u/v$, $z = u/v$, $\bar{z} = -z$, $z \in iR$ adevarat deoarece $AB \perp AC$, $u/v \in iR$ qed.

Metoda 6 : (vectoriala)

$\frac{AI}{DI} = \sqrt{2} = k$, $\overrightarrow{BI} = (\overrightarrow{BA} + \sqrt{2}\,\overrightarrow{BD}) / (1+\sqrt{2})$

$\overrightarrow{BI} = (\sqrt{2}-1)\overrightarrow{BA} + (2-\sqrt{2})\overrightarrow{BD} = (\sqrt{2}-1)\overrightarrow{BA} + \overrightarrow{BC}(2-\sqrt{2})/2$

Dar $\overrightarrow{AI} = \overrightarrow{AB} + \overrightarrow{BI} = (2-\sqrt{2})\overrightarrow{AB} - \overrightarrow{CB}(2-\sqrt{2})/2$

$\overrightarrow{AI}^2 = (2-\sqrt{2})^2(\overrightarrow{AB}^2 - \overrightarrow{AB}\,\overrightarrow{CB} + \overrightarrow{CB}^2/4)$

Dar $\overrightarrow{AB}\,\overrightarrow{CB} = |\overrightarrow{AB}||\overrightarrow{CB}|\cos(\angle B) = a\,a\sqrt{2}\,\sqrt{2}/2 = a^2$

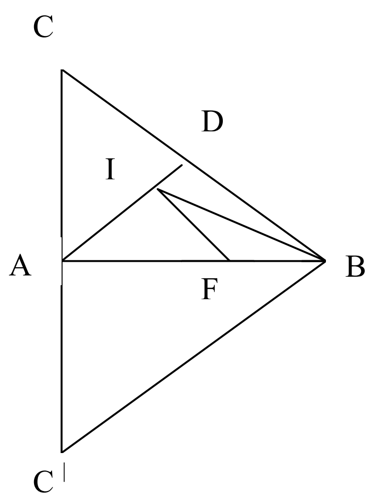
$\overrightarrow{AI}^2 = (6-4\sqrt{2})(a^2 - a^2 + 2a^2/4) = (3-2\sqrt{2})a^2 = (a\sqrt{2} - a)^2$ deci

$AI = a\sqrt{2} - a = BC - AB$, $AI + AB = BC$ sau $AI = -a\sqrt{2} + a = AB - BC < 0$ fals .

Metoda 7: (sintetica)

Construim $AC = AC^{|}$, puctele A , C, $C^{|}$ coliniare . Duc IF || BD , m($\angle$DBI)=m($\angle$BIF) (alt . int) deci ΔBIF isoscel adica BF=IF . Dar ΔAIF este dreptunghic isoscel deoarece m($\angle$IAF)=45^0 atunci AI= BF=IF . Calculez aria (in doua moduri) A $\Delta BCC^{|} = 2a^2/2 = a^2 = BC^2/2$, $BC^2 = 2a^2$. AB=a Dar $\Delta AIF \approx \Delta ADB$ (U.U) , $\frac{AF}{AB} = \frac{IF}{BD}$, $\frac{AB - AI}{AB} = \frac{2AI}{BC}$

Se gaseste ca $AI = \frac{AB * BC}{2AB + BC}$, inlocuim in relatia AI +AB =BC si obtinem ca

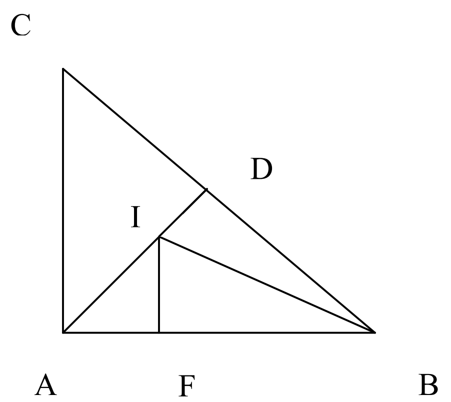


$\frac{AB * BC}{2AB + BC} + AB = BC$,

$2AB^2 + 2AB*BC = 2AB*BC + BC^2$,

$BC^2 = 2AB^2$ adevarat.

Metoda 8: (la nivelul clasei a 6-a)



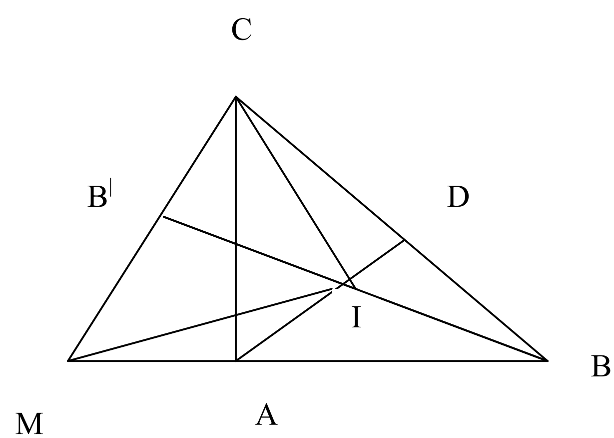
Duc IF $\perp$ AB rezulta ca ΔFIB $\equiv$ ΔDIB , BI lat . com si , m($\angle$DBI)=m($\angle$IBF) (I.U) . Dar ΔAIF este dreptunghic isoscel deoarece m($\angle$IAF)=45^0 , AF=IF =DI ,

AF+FB =AB , , DI + $\frac{BC}{2}$=AB , DI =AB - $\frac{BC}{2}$, AD=AI +DI , $\frac{BC}{2}$=AI +AB - $\frac{BC}{2}$

ΔABC isoscel AD inaltime deci AD este mediana deci AD =$\frac{BC}{2}$, rezulta ca

AI +AB=BC qed.

Metoda 9: (la nivelul clasei a 6-a)



Aleg punctul M a.i BM=BC si punctele M, A , B coliniare . ΔBMC isoscel , $BB^{|}$ bisectoare deci v-a fi si mediana. Dar AI +AB=BC , AI=BM-AB=MA . Trebuie sa arat ca AI=AM. Observ ca IC=IM deoarece Δ $B^{|}IC \equiv$ Δ $B^{|}IM$ si BI=CI deoarece Δ DIC$\equiv$ Δ DIB . Deci CI=MI=BI (I este centrul cercului circumscris Δ CMB) rezulta ca MI=BI , m($\angle$MAI)=180^0 -45^0=135^0 . Δ MIB isoscel , MI=BI , m($\angle$IBM)=m($\angle$IMB)=$22^0 30^{|}$, In Δ MAI , m($\angle$MAI)= 135^0 , m($\angle$IMA)= $22^0 30^{|}$ m($\angle$MIA)= $22^0 30^{|}$, AI=AM deci AI+AB=BC qed.

Metoda 10 : (trigonometrica)

In Δ AIB aplic teorema sinusurilor $\frac{AI}{\sin(\angle\frac{B}{2})}=\frac{AB}{\sin(180-45-\frac{B}{2})}=\frac{AB}{\sin(45+\frac{B}{2})}$

AI=$\frac{AB\sin(\angle\frac{B}{2})}{\sin(45+\frac{B}{2})}=\frac{a\sin(\angle\frac{B}{2})\sqrt{2}}{\sin(\angle\frac{B}{2})+\cos(\angle\frac{B}{2})}$. Aplic formulele: sin $(\frac{x}{2})=\sqrt{\frac{1-\cos x}{2}}$ si

Cos$(\frac{x}{2})=\sqrt{\frac{1+\cos x}{2}}$ si x=45^0 , $\sin(\angle\frac{B}{2})=\frac{\sqrt{2-\sqrt{2}}}{2}$,$\cos(\angle\frac{B}{2})=\frac{\sqrt{2+\sqrt{2}}}{2}$

$$AI=\frac{2a\sqrt{2-\sqrt{2}}}{\sqrt{2}*(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}})}=\frac{a\sqrt{2}*\sqrt{2-\sqrt{2}}(\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}})}{2+\sqrt{2}-2+\sqrt{2}}=$$

$\frac{a\sqrt{2}(\sqrt{2}-2+\sqrt{2})}{2\sqrt{2}}=a\sqrt{2}-a$ =BC-AB , AI+AB=BC qed.

Reciproca 1 :

"Se considera triunghiul ABC cu AI +AB=BC si m($\angle BAC$)=90^0 . Fie D∈ (BC) astfel incat AD ⊥ BC .Bisectoarea unghiului ABC intersecteaza dreapta AD in punctul I . Demonstrati ca AB=AC" .

Solutie: In Δ ABD aplic teorema bisectoarei

$\frac{AI}{AB}=\frac{DI}{DB}=tg(\angle\frac{B}{2})=\frac{BC-AB}{AB}=\frac{BC}{AB}-1=\frac{1}{\cos(\angle B)}-1$, folosim formula

tg(X / 2)= (sinX) / (1+cos X) , obtinem ca $\frac{\sin(\angle B)}{1+\cos(\angle B)}=\frac{1-\cos(\angle B)}{\cos(\angle B)}$,

$\sin(\angle B)\cos(\angle B)=\sin(\angle B)\sin(\angle B)$, $\sin(\angle B)(\cos(\angle B)-\sin(\angle B))=0$, dar

$\sin(\angle B)\neq 0$ deci $\sin(\angle B)=\cos(\angle B)$, unghiul B fiind ascutit rezulta ca

$m(\angle B)=45^0=m(\angle C)$, Δ ABC isoscel , AB=AC qed.

Reciproca 2 :

"Se considera triunghiul ABC ,AB=AC si m($\angle BAC$)=90^0 . Fie D∈ (BC) astfel incat bisectoarea unghiului ABC intersecteaza dreapta AD in punctul I cu AI +AB=BC. Demonstrati ca AD ⊥ BC" .

Solutie:In Δ ABD aplic teorema cosinisului $AD^2=AB^2+BD^2-2AB*BD*\cos 45^0$,

$AD^2=AB^2+BD^2-AB*BD*\sqrt{2}$, In Δ ABD aplic teorema bisectoarei $\frac{AI}{DI}=\frac{AB}{BD}$.

$\frac{AI}{AI+DI}=\frac{AB}{AB+BD}$, $\frac{AI}{AD}=\frac{AB}{AB+BD}$, AI=BC-AB= a√2 – a , AB=AC=a , BC=a√2 ,

$(\frac{AI}{AD})^2=(\frac{AB}{AB+BD})^2$, $[a^2(\sqrt{2}-1)^2]/[a^2+BD^2-a*BD*\sqrt{2}]=a^2/(a+BD)^2$,

$(3-2\sqrt{2})(a^2+2a\,BD+BD^2)=a^2+BD^2-a\,BD\sqrt{2}$, dupa efectuarea calculelor se obtine ca

$(2-2\sqrt{2})\,BD^2 + (6a-3\sqrt{2}a)\,BD + (2a^2 - 2a^2\sqrt{2}) = 0$, ecuatie de gradul doi in necunoscuta BD. Se gaseste $\Delta = a^2(2-\sqrt{2})^2$, ecuatia are doua solutii distincte
$BD = a\sqrt{2}/2$ rezulta ca $DC=BC-BD= a\sqrt{2}/2$, D este mijlocul lui [BC] , ΔABC isoscel AD mediana deci AD este inaltime adica $AD \perp BC$ qed. Si a doua solutie se obtine $BD= a\sqrt{2}=BC$ rezulta ca D=C fals .

Reciproca 3 :
"Se considera triunghiul ABC cu AI +AB=BC si AB=AC. Fie D∈ (BC) astfel incat $AD \perp BC$ .Bisectoarea unghiului ABC intersecteaza dreapta AD in punctul I . Demonstrati ca $m(\angle BAC)=90^0$" .

Solutie : In ΔABD aplic teorema bisectoarei $\frac{AI}{DI}=\frac{AB}{BD}=\frac{AI}{DI}=\frac{2AB}{BC}$. Deci

$\frac{AI}{AI+DI}=\frac{2AB}{BC+2AB}$, $\frac{BC-AB}{AD}=\frac{2AB}{BC+2AB}$, $AD=\frac{(BC-AB)(BC+2AB)}{2AB}$

In ΔABD aplic Teorema lui Pitagora $AD^2 + BD^2 = AB^2$, inlocuim in aceasta relatie pe AD si obtinem $\left(\frac{(BC-AB)(BC+2AB)}{2AB}\right)^2 + \left(\frac{BC}{2}\right)^2 = AB^2$. Dupa efectuarea calculelor se obtine ca $(BC^2 - 2AB^2)(BC^2 + 2AB*BC)=0$

Daca $BC^2 - 2AB^2 = 0$ rezulta din reciproca teoremei lui Pitagora ca $m(\angle BAC)=90^0$

Daca $BC^2 + 2AB*BC = 0$ fals deoarece $BC^2 + 2AB*BC > 0$.