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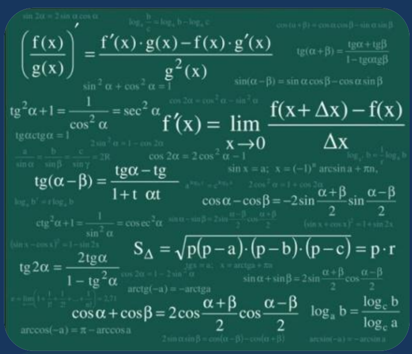

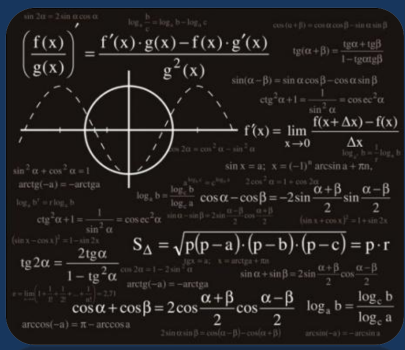

Articole:

1. Solutions of some problems from Octogon Mathematical Magazine

by D.M. Bătineţu-Giurgiu, National College ''Matei Basarab'', Bucharest, Romania
and
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PP.21151. In all triangle ABC holds:

1) $\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^2=\frac{3s^2-8r^2-32Rr}{s^2r^2}$;

2) $s^2\geq 2r^2+12Rr$.

Solution 1. We take in (*) from solution 2 of PP.21149: $x=\frac{1}{r_a}, y=\frac{1}{r_b}, z=\frac{1}{r_c}$, and we deduce (1). By Bergström's inequality we have

$$\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^2\geq\frac{1}{3}\left(\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)\right)^2=\frac{1}{3r^2},$$

and from (1) yields (2), and we are done.

Solution 2. We have:

$$\left(\frac{1}{r}-\frac{2}{r_a}\right)^2=\left(\frac{s}{sr}-\frac{2(s-a)}{sr}\right)^2=\frac{4a^2-4as+s^2}{s^2r^2},$$

and from $\sum a^2=2(s^2-r^2-4Rr)$, respectively $4\sum as=8s^2$ yields (1).
The inequality (2) easily follows from the inequality (2) of PP.21150.

Solution 3. Denoting by F the area of triangle ABC, we have:

$$\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^2=\sum\left(\frac{s}{F}-\frac{2(s-a)}{F}\right)^2=\frac{1}{F^2}\sum(2a-s)^2=\frac{1}{s^2r^2}\sum(4a^2-4as+s^2)=$$

$$=\frac{1}{s^2r^2}(8s^2-8r^2-32Rr-8s^2+3s^2)=\frac{3s^2-8r^2-32Rr}{s^2r^2},$$ and we are done.

PP.21152. In all triangle ABC holds:

1) $\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)^3=\left(2+\frac{r}{2R}\right)^3-\frac{3s^2}{2R^2}$;

2) $\left(2+\frac{r}{2R}\right)^3\geq\frac{27s^2}{16R^2}$.

Solution. We have: If $x,y,z\in\mathbf{R}$, then:

(*) $(-x+y+z)^3+(x-y+z)^3+(x+y-z)^3=(x+y+z)^3-24xyz$.

We take in (*) $x=\cos^2\frac{A}{2}, y=\cos^2\frac{B}{2}, z=\cos^2\frac{C}{2}$, and after some algebra we obtain (1). Also we have:

$\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)^3\geq\frac{1}{9}\left(\sum\left(2+\frac{r}{2R}-2\cos^2\frac{A}{2}\right)\right)^3=\frac{1}{9}\left(2+\frac{r}{2R}\right)^3$, and from (1)

$\left(2+\frac{r}{2R}\right)^3\geq\frac{27s^2}{16R^2}$, and we are done.

PP.21153. In all triangle ABC holds:

1) $\sum(s-a)^2=s^2-2r^2-4Rr$;

2) $s^2\geq3r^2+4Rr$.

Solution. We take in (*) from solution 2 of PP.21149: $x=a, y=b, z=c$ and we obtain (1). By Bergström's inequality we have: $\sum(s-a)^2\geq\frac{1}{3}\left(\sum(s-a)\right)^2=\frac{s^2}{3}$, and from (1) we deduce (2) and we are done.

PP.21154. In all triangle ABC holds $\sum(3a-b-c)^2=16(s^2-2r^2-4Rr)$.

Solution 1. The identity is not true. In fact we have the following identity:

$\sum(3a-b-c)^2=\sum(9a^2+b^2+c^2-6ab-6ac+2bc)=11\sum a^2-10\sum ab=$

$=11\left(\sum a\right)^2-22\sum ab-10\sum ab=44s^2-32(s^2+r^2+4Rr)=4(3s^2-8r^2-32Rr)$,

and we are done.

Solution 2. We take in (*) from solution 2 of PP.21149: $x=s-a, y=s-b, z=s-c$.

PP.21155. In all triangle ABC holds:

1) $\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^3=\frac{s^2-24r^2}{s^2r^3}$;

2) $s\geq3\sqrt{3}r$.

Solution 1. We take in (*) from solution to PP.21152: $x=\frac{1}{r_a}, y=\frac{1}{r_b}, z=\frac{1}{r_c}$, and after some algebra we obtain 1). We have: $\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^3 \geq \frac{1}{9}\left(\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)\right)^3=\frac{1}{9r^3}$, so by (1) yields that $\frac{s^2-24r^2}{s^2r^3}\geq\frac{1}{9r^3}$, and finnaly $s\geq 3\sqrt{3}r$, and we are done.

Solution 2. We have $\left(\frac{1}{r}-\frac{2}{r_a}\right)^3=\left(\frac{s-2(s-a)}{sr}\right)^3=\frac{(2a-s)^3}{s^3r^3}$. Also we have the relations:

$\sum a^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca), \sum a^2=2(s^2-r^2-4Rr)$,

$\sum ab=s^2+r^2+4rR$. By above we obtain:

$\sum(2a-s)^3=8\sum a^3-12s\sum a^2+6s^2\sum a-3s^3=$

$=8[2s(2s^2-2r^2-8Rr-s^2-r^2-4Rr)+3abc]-24s(s^2-r^2-4Rr)+9s^3=$

$=s(s^2-24r^2)$, so $\sum\left(\frac{1}{r}-\frac{2}{r_a}\right)^3=\frac{s^2-24r^2}{s^2r^3}$, and (1) is proved.

(2) is the item 5.11 ($s^2\geq 27r^2$) from Bottema.

PP.21156. In all triangle ABC holds:

1) $\sum\left(1-\frac{r}{2R}-2\sin^2\frac{A}{2}\right)^3=\left(1-\frac{r}{2R}\right)^3-\frac{3r^2}{2R^2}$;

2) $\left(1-\frac{r}{2R}\right)^3\geq\frac{27r^2}{16R^2}$.

Solution. We take in (*) from solution to PP.21152 $x=\sin^2\frac{A}{2}, y=\sin^2\frac{B}{2}, z=\sin^2\frac{C}{2}$ and after some algebra we obtain (1). We have:

$\sum\left(1-\frac{r}{2R}-2\sin^2\frac{A}{2}\right)^3\geq\frac{1}{9}\left(\sum\left(1-\frac{r}{2R}-2\sin^2\frac{A}{2}\right)\right)^3=\frac{1}{9}\left(1-\frac{r}{2R}\right)^3$, and by (1) we deduce that $\left(1-\frac{r}{2R}\right)^3\geq\frac{27r^2}{16R^2}$, and the proof is complete.

PP.21157. In all triangle ABC holds:

1) $s^3=(s-a)^3+(s-b)^3+(s-c)^3+12sRr$;

2) $2s^2\geq 27Rr$.

Solution 1. We take in (*) from solution to PP.21152: $x=a, y=b, z=c$ and after some algebra yields (1), than by

$\frac{s^3-12sRr}{3}=\frac{\sum(s-a)^3}{3}\geq\left(\frac{1}{3}\sum(s-a)\right)^3=\frac{s^3}{27}$, follows (2), and we are done.

Solution 2. We have:

$$s^3=(s-a+s-b+s-c)^3=(s-a)^3+(s-b)^3+(s-c)^3+$$
$$+3(s-a)(s-b)(s-a+s-b)+3(s-b)(s-c)(s-b+s-c)+$$
$$+3(s-a)(s-c)(s-a+s-c)+6(s-a)(s-b)(s-c)=\sum(s-a)^3+$$
$$+3\sum(as^2-abs-acs+abc)+6\cdot\frac{s(s-a)(s-b)(s-c)}{s}=$$
$$=\sum(s-a)^3+6s^3-6s\sum ab+9abc+6sr^2=$$
$$=\sum(s-a)^3+6s^3-6s^3-6sr^2-24sRr+36sRr+6sr^2=$$

$=(s-a)^3+(s-b)^3+(s-c)^3+12sRr$, so (1) is proved.

The inequality (2) is the item 5.12 from Bottema. The proof is complete.

PP.21158. In all triangle ABC holds $\sum(-a+3b-c)^3=8s(s^2-24r^2)$.

Solution 1. We have $ab+bc+ca=s^2+r^2+4Rr$, $abc=4Rrs$,

$$a^3+b^3+c^3-3abc=(a+b+c)[(a+b+c)^2-3(ab+bc+ca)]=$$
$$=2s(s^2-3r^2-12Rr).$$

So, $\sum(-a+3b-c)^3=\sum(4b-2s)^3=8\sum(2b-s)^3=$

$$=8\cdot[8\sum a^3-24abc+24abc-12s\sum a^2+6s^2\sum a-3s^3]=$$
$$=8\cdot[16s(s^2-3r^2-12Rr)+96Rrs-12s\left(\sum a\right)^2+24s\sum ab+12s^3-3s^3]=$$
$$=8s\cdot[16s^2-48r^2-192Rr+96Rr-48s^2+24s^2+24r^2+96Rr+9s^2]=$$

$=8s(s^2-24r^2)$, and we are done.

Solution 2. We take in (*) from solution to PP.21152: $x=s-a, y=s-b, z=s-c$.

PP.21159. In all triangle ABC holds:

1) $\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)^3=\frac{s^2-12Rr}{s^2r^3}$;

2) $2s^2\geq 27Rr$.

Solution 1. We take in (*) from solution to PP.21152 $x=\frac{1}{h_a}, y=\frac{1}{h_b}, z=\frac{1}{h_c}$ and after some algebra we obtain (1). We have:

$\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)^3 \geq \frac{1}{9}\left(\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)\right)^3 = \frac{1}{sr^3}$, so $\frac{s^2-12Rr}{s^2r^3} \geq \frac{1}{9r^3}$, which is equivalent with (2).

Solution 2. Denoting by F the area of triangle ABC by PP.1157 we have:

$\sum(s-a)^3 = s^3-12sRr$. So,

$$\sum\left(\frac{1}{r}-\frac{2}{h_a}\right)^3 = \sum\left(\frac{s}{F}-\frac{a}{F}\right)^3 = \frac{1}{F^3}\sum(s-a)^3 = \frac{s(s^2-12Rr)}{s^3r^3} = \frac{s^2-12Rr}{s^2r^3}.$$

The inequality, (2), i.e. $2s^2 \geq 27Rr$ is the item 5.12 from Bottema, and we are done.

PP.21160. In all triangle ABC holds $\left(\sum a^2\right)^2 \leq 2\sum a^4 + \frac{16s^4}{27}$.

Solution. Because $\left(\sum a\right)^4 = \sum a^4 + 4\sum a^3b + 4\sum ab^3 + 6\sum a^2b^2 + 12\sum a^2bc$, the inequality from the statement becomes:

$27\sum a^4 + 54\sum a^2b^2 \leq 54\sum a^4 + \sum a^4 + 4\sum a^3b + 4\sum ab^3 + 6\sum a^2b^2 + 12\sum a^2bc$

$\Leftrightarrow 7\sum a^4 + \sum a^3b + \sum ab^3 + 3\sum a^2bc \geq 12\sum a^2b^2$ (1)

Applying Schur's inequality we obtain:

$\sum a^2(a-b)(a-c) \geq 0 \Leftrightarrow \sum a^4 + \sum a^2bc \geq \sum a^3b + \sum ab^3$, and with AM-GM inequality we deduce $\sum a^3b + \sum ab^3 \geq 2\sum a^2b^2$, so $\sum a^4 + \sum a^2bc \geq 2\sum a^2b^2$.

We use also the inequality $\sum a^4 \geq \sum a^2b^2$.

By above yields that:

$7\sum a^4 + \sum a^3b + \sum ab^3 + 3\sum a^2bc = 3\left(\sum a^4 + \sum a^2bc\right) + \sum a^3b + \sum ab^3 + 4\sum a^4 \geq$

$\geq 6\sum a^2b^2 + 2\sum a^2b^2 + 4\sum a^2b^2 = 12\sum a^2b^2$, and the proof is complete.

PP.21163. Prove that $(x+y)^{2n+1}-x^{2n+1}-y^{2n+1}$ is divisible by $xy(x+y)$, for all $n \in N$.

Solution. We have:

$x^{2n+1}+y^{2n+1} = (x+y)(x^{2n}-x^{2n-1}y+...-xy^{2n-1}+y^{2n})$, so

$(x+y)^{2n+1}-x^{2n+1}-y^{2n+1}$ is divisible by $x+y$ (1)

On the other hand, he have:

$(x+y)^{2n+1}-x^{2n+1}-y^{2n+1} =$

$$=\binom{2n+1}{1}x^{2n}y+\binom{2n+1}{2}x^{2n-1}y^2+...+\binom{2n+1}{2n-1}xy^{2n-1}+\binom{2n+1}{2n}xy^{2n} \qquad (2)$$

From (1) and (2) follows the desired result.

PP.21164. In all triangle ABC holds $\sum ctg\frac{A}{2}\left(1-ctg^2\frac{B}{2}\right)\left(1-ctg^2\frac{C}{2}\right)=\frac{4s}{r}$.

Solution. It well-known that $ctg\frac{A}{2}=\frac{s-a}{r}$ and $\sum ctg\frac{A}{2}=\prod ctg\frac{A}{2}$. We obtain

$$\sum ctg\frac{A}{2}=\frac{(s-a)(s-b)(s-c)}{r^3}=\frac{s(s-a)(s-b)(s-c)}{sr^3}=\frac{s^2r^2}{sr^3}=\frac{s}{r};$$

$$\sum\left(ctg^2\frac{A}{2}ctg\frac{B}{2}+ctg\frac{A}{2}ctg^2\frac{B}{2}\right)=\sum\frac{(s-a)(s-b)}{r^2}\cdot\frac{s-a+s-b}{r}=$$

$$=\frac{1}{r^3}\sum c(s-a)(s-b)=\frac{1}{r^3}\sum(s^2c-acs-bcs+abc)=$$

$$=\frac{1}{r^3}(2s^3-2s\sum ab+3abc)=\frac{1}{r^3}(2s^3-2s^3-2sr^2-8Rrs+12Rrs)=\frac{4Rs-2rs}{r^2},$$

and $\sum ctg\frac{A}{2}ctg\frac{B}{2}=\sum\frac{(s-a)(s-b)}{r^2}=\frac{1}{r^2}(3s^2-4s^2+\sum ab)=\frac{r+4R}{r}$.

Yields that: $\sum ctg\frac{A}{2}\left(1-ctg^2\frac{B}{2}\right)\left(1-ctg^2\frac{C}{2}\right)=$

$$=\sum ctg\frac{A}{2}-\sum\left(ctg^2\frac{A}{2}ctg\frac{B}{2}+ctg\frac{A}{2}ctg^2\frac{B}{2}\right)+\prod ctg\frac{A}{2}\sum ctg\frac{A}{2}ctg\frac{B}{2}=$$

$=\frac{s}{r}+\frac{2rs-4Rs}{r^2}+\frac{s}{r}\cdot\frac{r+4R}{r}=\frac{4s}{r}$, and the proof is complete.

PP.21165. If $x,y,z>0$, then $\prod\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}\geq\frac{343}{125}\left(\sum xy\right)^3$.

Solution. After some algebra we obtain that:

$\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}=\frac{7}{5}(x^2+xy+y^2)$, so the inequality to prove is equivalent, successively with:

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\geq(xy+yz+zx)^2 \qquad (*)$$

$$\Leftrightarrow\sum x^4y^2+\sum x^2y^4+3x^2y^2z^2+2\sum x^3y^2z+2\sum x^3yz^2+\sum x^3y^3+\sum x^4yz\geq$$

$$\geq\sum x^3y^3+3\sum x^3y^2z+3\sum x^3yz^2+6x^2y^2z^2$$

$$\Leftrightarrow\sum x^4yz+\sum x^4y^2+\sum x^2y^4\geq\sum x^3y^2z+\sum x^3yz^2+3x^2y^2z^2.$$

Because $\sum x^4yz\geq 3x^2y^2z^2$, it remains to show that:

$$\sum x^4y^2 + \sum x^2y^4 \geq \sum x^3y^2z + \sum x^3yz^2 \qquad (1)$$

But $(4,2,0) \succ (3,2,1)$ and applying Muirhead's inequality, we obtain:

$$\sum_{sym} x^4y^2 \geq \sum_{sym} x^3y^2z$$, i.e. exactly (1).

Other proof for (1) is with the AM-GM inequality. Indeed, we have that:

$x^4y^2 + x^2z^4 \geq 2x^3yz^2$; $y^4z^2 + x^4y^2 \geq 2x^2y^3z$; $z^4x^2 + z^2y^4 \geq 2z^3xy^2$;

$y^4x^2 + z^2x^4 \geq 2x^3y^2z$; $z^4y^2 + y^4x^2 \geq 2z^2y^3x$; $x^4z^2 + y^2z^4 \geq 2z^3yx^2$,

which by adding up yields (1), and the proof is complete.

PP.21166. Prove that $(x+y+z)^{2n+1} - x^{2n+1} - y^{2n+1} - z^{2n+1}$ is divisible by $(x+y)(y+z)(z+x)$ for all $n \in N$.

Solution. Because $a^n - b^n$ is divisible by $a-b$ and $a^{2n+1} + b^{2n+1}$ is divisible by $a+b$, we have: $(x+y+z)^{2n+1} - x^{2n+1}$ is divisible by $x+y+z-x = y+z$ and $y^{2n+1} + z^{2n+1}$ is divisible by $y+z$, so $(x+y+z)^{2n+1} - x^{2n+1} - y^{2n+1} - z^{2n+1}$ is divisible by $y+z$ and similarly $(x+y+z)^{2n+1} - x^{2n+1} - y^{2n+1} - z^{2n+1}$ is divisible by $x+y$, respectively $(x+y+z)^{2n+1} - x^{2n+1} - y^{2n+1} - z^{2n+1}$ is divisible by $z+x$, and the conclusion follows.

2. Other solutions from some problems from The Pentagon journal

By Nela Ciceu, Roşiori, Bacău, Romania

And

Roxana Mihaela Stanciu, Buzău, Romania

Problem 722. *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.*

Characterize those positive integers n for which $2^{n^2}+1$ may be a prime number.

Solution:

For $n=1$, $2^1+1=3$ is prime number.

We shall prove that if n is not a power of 2, then $2^{n^2}+1$ is composite.

Let $n=2^t\cdot s$, where $t\geq 0$ and $s>1$ is odd.

We have that:

$$2^{n^2}+1=\left(2^{2^t}\right)^s+1=\left(\left(2^{2^t}+1\right)-1\right)^s+1=M\left(2^{2^t}+1\right)+(-1)^s+1=$$
$$=M\left(2^{2^t}+1\right)$$

Yields that a necessary but not sufficient for that

$$2^{n^2}+1$$

to be prime number is that $n=2^t$, with $t\geq 0$.

Problem 723. *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.*

Prove that there are infinitely many primitive Pythagorean triples (a, b, c), like $(5, 12, 13)$, with hypotenuse c such that the odd leg is a pentagonal number and the even leg is consecutive with the hypotenuse.

Solution:

A primitive Pythagorean triples is on form

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2,$$

where m and n are coprime, with different parity and $m > n$. Because b and c are consecutive we have $m^2 + n^2 = 2mn + 1 \Leftrightarrow (m-n)^2 = 1$, so $m = n + 1$. The odd leg i.e. $a = m^2 - n^2$ must be pentagonal n umber , i.e. we must find k such that

$$2n + 1 = \frac{k(3k-1)}{2} \Leftrightarrow n = \frac{3k^2 - k - 2}{4}.$$

Because n is positive integer we can take $k = 4k + 2$, where t is positive integer.
We obtain:

$$n = 12t^2 + 11t + 2 \text{ and } m = 12t^2 + 11t + 3.$$

The numbers m, n are consecutive, so is coprime and with different parity.
Therefore, yields the following family of Pythagorean primitive triangles with the properties of enunciation:

$$a = \frac{(4t+2)[3(4t+2)-1]}{2}, \; b = 2(12t^2 + 11t + 2)(12t^2 + 11t = 3),$$

$$c = 2(12t^2 + 11t + 2)(12t^2 + 11t + 3) + 1.$$

Remark. We can take also $k = 4t + 1$, and we obtain other family of triangles with $n = 12t^2 + 5t$ and $m = 12t^2 + 5t + 1$.
The proof is complete.

Problem 724. *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.*

Let $T_n = \frac{n(n+1)}{2}$ be the nth triangular number. Prove that the fraction

$$\frac{T_2 T_4 T_6 \cdots T_{2n}}{T_1 T_3 T_5 \cdots T_{2n-1}}$$

is always an integer.

Solution:

We have:

$$\frac{T_2T_4...T_{2n}}{T_1T_3...T_{2n-1}} = \frac{2\cdot 3\cdot 4\cdot 5\cdot ...\cdot (2n)(2n+1)}{1\cdot 2\cdot 3\cdot 4\cdot ...\cdot (2n-1)(2n)} = 2n+1$$, and we are done.

Problem 725. *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.*

It is known that each integer $n > 11$ is the sum of two composite numbers but the usual proof of this uses two different expressions, one for n even and one for n odd. If we restrict our attention to certain sequences of the natural numbers, then we can find one expression for each of the numbers in the sequence as a sum of two composite numbers, regardless of parity. Do this for the squares greater than 9 and the triangular numbers greater than 10.

Solution:

We have:

- $n^2 = 4+(n-2)(n+2)$; for $n \geq 4$, the number $(n-2)(n+2)$ is composite
- $\frac{n(n+1)}{2} = 6 + \frac{(n-3)(n+4)}{2}$; for $n \geq 5$, $n-3$ and $n+4$ have different parity so the number $\frac{(n-3)(n+4)}{2}$ is composite.

The proof is complete

Problem 726. *Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.*

Let x, y, and z be positive real numbers. Prove that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{xy}{y+z}+\frac{yz}{z+x}+\frac{zx}{x+y}\right) \geq \frac{9}{2}.$$

Solution:

Solution 1. We denotes: $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$ with $a,b,c>0$ the inequality of the enunciation becomes:

$$LHS = (a+b+c)\sum_{cyclic}\frac{c}{a(b+c)} \geq \frac{9}{2} \quad (1)$$

We have:

$$LHS=(a+b+c)\sum_{cyclic}\frac{c}{a(b+c)}=\sum_{cyclic}\frac{c}{b+c}+\sum_{cyclic}\frac{bc}{a(b+c)}+\sum_{cyclic}\frac{c^2}{ab+ac}=$$

$$=\sum_{cyclic}\frac{c^2}{ab+ac}+\sum_{cyclci}\frac{ac+bc}{a(b+c)}=\sum_{cyclic}\frac{c^2}{ab+ac}+\sum_{cyclic}\frac{c(a+b)}{a(b+c)} \qquad (2)$$

By the inequality of Harald Bergström we deduce that:

$$\sum_{cyclic}\frac{c^2}{ab+ac}\geq\frac{(a+b+c)^2}{\sum_{cyclic}(ab+ac)}=\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2},$$

so from (2) we obtain that:

$$LHS\geq\frac{3}{2}+\sum_{cyclic}\frac{c(a+b)}{a(b+c)}\overset{AM-GM}{\geq}\frac{3}{2}+3\sqrt[3]{\frac{c(a+b)}{a(b+c)}\cdot\frac{a(b+c)}{b(c+a)}\cdot\frac{b(c+a)}{c(a+b)}}=\frac{3}{2}+3=\frac{9}{2}\text{, Q.E.D.}$$

Solution 2. We have:

$$LHS=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\sum_{cyclic}\frac{xy}{y+z}=\sum_{cyclic}\frac{y}{y+z}+\sum_{cyclic}\frac{x}{y+z}+\sum_{cyclic}\frac{xy}{z(y+z)}\overset{Nesbitt}{\geq}$$

$$\overset{Nesbitt}{\geq}\frac{3}{2}+\sum_{cyclic}\frac{xy+yz}{z(y+z)}=\frac{3}{2}+\sum_{cyclic}\frac{y(x+z)}{z(y+z)}\overset{AM-GM}{\geq}\frac{3}{2}+3\sqrt[3]{\frac{y(x+z)}{z(y+z)}\cdot\frac{z(y+x)}{x(z+x)}\cdot\frac{x(z+y)}{y(x+y)}}=$$

$$=\frac{3}{2}+3=\frac{9}{2}\text{, Q.E.D.}$$

Problem 727. *Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.*

Let α, β, γ be the measure of the angles of a triangle ABC. Prove that

$$\sum_{cyclic}\frac{\sin\alpha}{4\sin\beta+5\sqrt{\sin\alpha\sin\beta}}\geq\frac{1}{3}.$$

Solution:

We have:

$$LHS=\sum_{cyclic}\frac{\sin\alpha}{4\sin\beta+5\sqrt{\sin\alpha\sin\beta}}\overset{AM-GM}{\geq}\sum_{cyclic}\frac{2\sin\alpha}{8\sin\beta+5(\sin\alpha+\sin\beta)}=$$

$$=2\sum_{cyclic}\frac{\sin\alpha}{13\sin\beta+5\sin\alpha}=2\sum_{cyclic}\frac{\sin^2\alpha}{5\sin^2\alpha+13\sin\alpha\sin\beta}\overset{Bergstrom}{\geq}$$

$$\overset{Bergstrom}{\geq}2\cdot\frac{(\sin\alpha+\sin\beta+\sin\gamma)^2}{\sum_{cyclic}(5\sin^2\alpha+13\sin\alpha\sin\beta)}=$$

$$= 2 \cdot \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{5(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) + 13(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha)} =$$

$$= 2 \cdot \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{5(\sin\alpha + \sin\beta + \sin\gamma)^2 + 3(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha)},$$

and because

$$(\sin\alpha + \sin\beta + \sin\gamma)^2 \geq 3(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha),$$

we obtain that:

$$U \geq 2 \cdot \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{5(\sin\alpha + \sin\beta + \sin\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2} = \frac{2(\sin\alpha + \sin\beta + \sin\gamma)^2}{6(\sin\alpha + \sin\beta + \sin\gamma)^2} = \frac{1}{3}, \text{ Q.E.D.}$$

3.Inegalităţi cu radicali

Prof. Ciobîcă C Constantin, Colegiul Vasile Lovinescu , Fălticeni

1. Să se demonstreze inegalitatea:
$\sqrt{3n+2m+1}+\sqrt{5n+2m+3}\le 2\cdot\sqrt{4n+2m+2};\ \forall n\in N,\forall m\in N$

Rezolvare:
Ridicăm la puterea a doua şi obţinem:
$3n+2m+1+5n+2m+3+2\sqrt{(3n+2m+1)\cdot(5n+2m+3)}\le 16n+8m+8$
$2\cdot\sqrt{(3n+2m+1)\cdot(5n+2m+3)}\le 8n+4m+4\Rightarrow$
$\sqrt{(3n+2m+1)(5n+2m+3)}\le 4n+2m+2$
Ridicăm la puterea a doua şi obţinem:
$(3n+2m+1)\cdot(5n+2m+3)\le(4n+2m+2)^2$
$15n^2+6nm+9n+10nm+4m^2+6m+5n+2m+3\le 16n^2+4m^2+4+16nm+16n+8m$
$n^2+2n+1\ge 0\Rightarrow(n+1)^2\ge 0;\forall n\in N$

2. Să se demonstreze inegalitatea: $\sqrt{(2p+1)n+5}+\sqrt{(2p+3)n+15}\le 2\sqrt{(2p+2)n+10};\ \forall n\in N,\forall p\in N$
Rezolvare:

Ridicăm la puterea a doua şi obţinem:
$2pn+n+5+2pn+3n+15+2\cdot\sqrt{[(2p+1)n+5]\cdot[(2p+3)n+15]}\le 8pn+8n+40$
$2\cdot\sqrt{[(2p+1)n+5]\cdot[(2p+3)n+15]}\le 4pn+4n+20\Rightarrow$
$\Rightarrow\sqrt{(2pn+n+5)\cdot(2pn+3n+15)}\le 2pn+2n+10$
Ridicăm la puterea a doua şi obţinem:
$(2pn+n+5)\cdot(2pn+3n+15)\le(2pn+2n+10)^2$
$4p^2n^2+6pn^2+30pn+2pn^2+3n^2+15n+10pn+15n+75\le$
$\le 4p^2n^2+4n^2+100+8pn^2+40pn+40n$
$n^2+10n+25\ge 0\Rightarrow(n+5)^2\ge 0,\forall n\in N$

3. Să se demonstreze inegalitatea: $\sqrt{an+p}+\sqrt{bn+q}\le 2\cdot\sqrt{\frac{a+b}{2}n+\frac{p+q}{2}};\forall n\in N,a,b,p,q\in R^*_+$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:
$an+p+bn+q+2\cdot\sqrt{(an+p)\cdot(bn+q)}\le 2(a+b)n+2p+2q$

$2\cdot\sqrt{(an+p)\cdot(bn+q)}\le an+bn+p+q$

Ridicăm la puterea a doua şi obţinem:

$4\cdot(an+p)\cdot(bn+q)\le(an+bn+p+q)^2\Rightarrow$

$(4an+4p)\cdot(bn+q)\le a^2n^2+b^2n^2+p^2+q^2+2abn^2+2anp+2anq+2bnp+2bnq+2pq$

$(4an+4p)\cdot(bn+q)=4an^2+4anq+4pbn+4pq$

$a^2n^2+b^2n^2+p^2+q^2-2abn^2+2anp-2anq-2bnp+2bnq-2pq\ge 0$

$(an-bn+p-q)^2\ge 0;\quad \forall n\in N, a,b,p,q\in R_+^*$

4. Să se demonstreze inegalitatea: $\sqrt{(2p+1)n+2m+1}+\sqrt{(2p+3)n+2m+3}\le 2\cdot\sqrt{(2p+2)n+2m+2}\quad ;\forall n\in N$ $,\forall p\in N,\forall m\in N$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$(2p+1)n+2m+1+(2p+3)n+2m+3+2\cdot\sqrt{(2p+1)n+2m+1}\cdot\sqrt{(2p+3)n+2m+3}\le$

$\le 4[(2p+2)n+2m+2]$

$2pn+n+2m+1+2pn+3n+2m+3+2\sqrt{[(2p+1)n+2m+1]\cdot[(2p+3)n+2m+3]}\le$

$\le 8pn+8n+8m+8$

$2\sqrt{[(2p+1)n+2m+1]\cdot[(2p+3)n+2m+3]}\le 4pn+4n+4m+4$

$\sqrt{[(2p+1)n+2m+1]\cdot[(2p+3)n+2m+3]}\le 2pn+2n+2m+2$

Ridicăm la puterea a doua şi obţinem:

$(2pn+n+2m+1)(2pn+3n+2m+3)\le(2pn+2n+2m+2)^2$

$4p^2n^2+6pn^2+4pnm+6pn+2pn^2+3n^2+2mn+3n+4pnm+3mn+4m^2+6m+2pn+$

$+3n+2n+3\le 4p^2n^2+4n^2+4m^2+4+8pn^2+8pnm+8pn+8nm+8n+8m$

$\Rightarrow n^2+2n+1\ge 0\Rightarrow(n+1)^2\ge 0;\forall n\in N$

5. Să se demonstreze inegalitatea: $n+\sqrt{3n^2+4}\le 2\cdot\sqrt{2n^2+2};\quad \forall n\in N$.

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$n^2+3n^2+4+2n\sqrt{3n^2+4}\le 8n^2+8$

$2n\sqrt{3n^2+4}\le 4n^2+4\Rightarrow n\sqrt{3n^2+4}\le 2n^2+2$

Ridicăm la puterea a doua şi obţinem:

$3n^4+4n^2\le 4n^4+8n^2+4\Rightarrow n^4+4n^2+4\ge 0\Rightarrow(n^2+2)^2\ge 0,\forall n\in N$

6. Să se demonstreze inegalitatea: $(p+1)n+\sqrt{(p+1)^2n^2+4}\le 2\cdot\sqrt{(p+1)^2n^2+2}\ ;\forall n\in N,\forall p\in N$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$$(p+1)^2 n^2 + (p+1)^2 n^2 + 4 + 2\cdot(p+1)n\sqrt{(p+1)^2 n^2 + 4} \le 4(p+1)^2 n^2 + 8$$

$$2(p+1)n\sqrt{(p+1)^2 n^2 + 4} \le 2(p+1)^2 n^2 + 4$$

$$(p+1)n\sqrt{(p+1)^2 n^2 + 4} \le (p+1)^2 n^2 + 2$$

Ridicăm la puterea a doua şi obţinem:

$$(p+1)^4 n^4 + 4(p+1)^2 n^2 \le (p+1)^4 n^4 + 4(p+1)^2 n^2 + 4 \Rightarrow 4 \ge 0; \forall n \in N, \forall p \in N$$

7. Să se demonstreze inegalitatea: $an + \sqrt{bn^2 + 1} \le \sqrt{(2a^2 + 2b)n^2 + 2} \quad ; \forall n \in N, a, b \in R_+^*$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$$a^2 n^2 + bn^2 + 1 + 2an\sqrt{bn^2 + 1} \le 2a^2 n^2 + 2bn^2 + 2$$

$$2an\sqrt{bn^2 + 1} \le a^2 n^2 + bn^2 + 1$$

Ridicăm la puterea a doua şi obţinem:

$$4a^2 n^2 (bn^2 + 1) \le a^4 n^4 + b^2 n^4 + 1 + 2a^2 n^4 b + 2a^2 n^2 + 2bn^2$$

$$0 \le a^4 n^4 + b^2 n^4 + 1 - 2a^2 n^4 b - 2a^2 n^2 + 2bn^2 \Rightarrow (a^2 n^2 - bn^2 - 1)^2 \ge 0; \forall n \in N, a, b \in R_+^*$$

8. Să se demonstreze inegalitatea $\sqrt{n^2 + 2a} + \sqrt{n^2 + 2b} \le 2\cdot\sqrt{n^2 + a + b}; \forall n \in N, a, b \in R_+^*$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$$n^2 + 2a + n^2 + 2b + 2\sqrt{(n^2 + 2a)(n^2 + 2b)} \le 4n^2 + 4a + 4b$$

$$\sqrt{(n^2 + 2a)(n^2 + 2b)} \le n^2 + a + b$$

Ridicăm la puterea a doua şi obţinem:

$$(n^2 + 2a)(n^2 + 2b) \le (n^2 + a + b)^2$$

$$n^4 + 2n^2 b + 2n^2 a + 4ab \le n^4 + a^2 + b^2 + 2n^2 a + 2n^2 b + 2ab$$

$$a^2 + b^2 - 2ab \ge 0 \Rightarrow (a-b)^2 \ge 0; \forall a, b \in R_+^*$$

9. Să se demonstreze inegalitatea: $\sqrt{2an^2 + 1} + \sqrt{2bn^2 + 3} \le 2\sqrt{(a+b)n^2 + 2}; \quad \forall n \in N, a, b \in R_+^*$

Rezolvare:

Ridicăm la puterea a doua şi obţinem:

$$2an^2 + 1 + 2bn^2 + 3 + 2\sqrt{(2an^2 + 1)(2bn^2 + 3)} \le 4an^2 + 4bn^2 + 8$$

$$\sqrt{4abn^4 + 2bn^2 + 6an^2 + 3} \le an^2 + bn^2 + 2$$

Ridicăm la puterea a doua şi obţinem:

$$4abn^4 + 2bn^2 + 6an^2 + 3 \le a^2 n^4 + b^2 n^4 + 4 + 2abn^4 + 4an^2 + 4bn^2$$

$$0 \le a^2 n^4 + b^2 n^4 - 2abn^4 + 2bn^2 - 2an^2 + 1 \Rightarrow (an^2 - bn^2 - 1)^2 \ge 0; \forall a, b \in R_+^*$$

10. Să se demonstreze inegalitatea:

$$\sqrt{2}+\sqrt{\frac{11}{2}}+\sqrt{11}+...+\sqrt{\frac{2n^2+n+1}{2}}\geq\frac{2n^2+3n}{4},\forall n\in N^*$$

Rezolvare:

$$\sqrt{2}+\sqrt{\frac{11}{2}}+\sqrt{11}+...+\sqrt{\frac{2n^2+n+1}{2}}=\sum_{k=1}^{n}\sqrt{\frac{2k^2+k+1}{2}}$$

$$\sqrt{\frac{2k^2+k+1}{2}}\geq k+\frac{1}{4}$$

Demonstrăm inegalitatea ridicând la puterea a doua:

$$\frac{2k^2+k+1}{2}\geq k^2+\frac{k}{2}+\frac{1}{16}\Big|\cdot 2$$

$$2k^2+k+1\geq 2k^2+k+\frac{1}{8}\Rightarrow 1\geq\frac{1}{8}(A)$$

$$\sum_{k=1}^{n}\sqrt{\frac{2k^2+k+1}{2}}\geq\sum_{k=1}^{n}\left(k+\frac{1}{4}\right)=1+\frac{1}{4}+2+\frac{1}{4}+...+n+\frac{1}{4}=1+2+...+n+\frac{n}{4}=$$

$$=\frac{n^2+n}{2}+\frac{n}{4}=\frac{2n^2+3n}{4}$$

4. Sume cu şiruri de numere reale în progresii geometrice (generalizări)

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1. Fie $(b_n)_{n\in N^*}\in R^*$ un şir de numere reale în progresie geometrică de raţie $q\neq 1$, atunci demonstraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n}(b_{2k+3}+b_{2k+1})\cdot\sum_{k=1}^{n}(b_{2k+5}+b_{2k+3})}+\frac{q^2}{\sum_{k+1}^{n}(b_{2k+5}+b_{2k+3})\cdot\sum_{k=1}^{n}(b_{2k+7}+b_{2k+5})}+$$

$$+\ldots+\frac{q^{2i}}{\sum_{k+1}^{n}(b_{2k+2i+3}+b_{2k+2i+1})\cdot\sum_{k=1}^{n}(b_{2k+2i+5}+b_{2k+2i+3})}=$$

$$=\frac{q^{2i+2}-1}{q^2-1}\cdot\left[\frac{1}{\sum_{k=1}^{n}(b_{2k+3}+b_{2k+1})}-\frac{1}{\sum_{k=1}^{n}(b_{2k+2i+5}+b_{2k+2i+3})}\right],\ \forall i\in N,\forall n\in N^*$$

Rezolvare:

$$\sum_{k=1}^{n}(b_{2k+5}+b_{2k+3})-\sum_{k=1}^{n}(b_{2k+3}+b_{2k+1})=\sum_{k=1}^{n}(b_{2k+5}-b_{2k+1})=\sum_{k=1}^{n}(b_1\cdot q^{2k+4}-b_1\cdot q^{2k})=$$

$$=\sum_{k=1}^{n}b_1\cdot q^{2k}\cdot(q^4-1)=b_1\cdot(q^4-1)\cdot(q^2+q^4+q^6+\ldots+q^{2n})=$$

$$=b_1\cdot(q^2-1)(q^2+1)\cdot q^2\cdot(1+q^2+q^4+\ldots+q^{2n-2})=$$

$$=b_1\cdot(q^2-1)\cdot(q^2+1)\cdot q^2\cdot\frac{q^{2n}-1}{q^2-1}=b_1\cdot(q^{2n}-1)\cdot(q^4+q^2)$$

$$\frac{1}{\sum_{k=1}^{n}(b_{2k+3}+b_{2k+1})\cdot\sum_{k=1}^{n}(b_{2k+5}+b_{2k+3})}=$$

$$=\frac{1}{b_1\cdot(q^{2n}-1)\cdot(q^4+q^2)}\cdot\left[\frac{1}{\sum_{k=1}^{n}(b_{2k+3}+b_{2k+1})}-\frac{1}{\sum_{k=1}^{n}(b_{2k+5}+b_{2k+3})}\right].$$

$$\sum_{k=1}^{n}(b_{2k+7}+b_{2k+5})-\sum_{k=1}^{n}(b_{2k+5}+b_{2k+3})=\sum_{k=1}^{n}(b_{2k+7}-b_{2k+3})=$$

$$=\sum_{k=1}^{n}\left(b_1\cdot q^{2k+6}-b_1\cdot q^{2k+2}\right)=b_1\cdot q^2\cdot\left(q^4-1\right)\cdot\left(q^2+q^4+q^6+...+q^{2n}\right)=$$

$$=b_1\cdot q^2\cdot\left(q^2-1\right)\cdot\left(q^2+1\right)\cdot q^2\cdot\left(1+q^2+q^4+...+q^{2n-2}\right)=$$

$$=b_1\cdot q^2\cdot\left(q^2-1\right)\cdot\left(q^2+1\right)\cdot q^2\cdot\frac{q^{2n}-1}{q^2-1}=b_1\cdot\left(q^4+q^2\right)\left(q^{2n-1}\right)\cdot q^2$$

$$\frac{q^2}{\sum_{k=1}^{n}\left(b_{2k+7}+b_{2k+5}\right)\cdot\sum_{k=1}^{n}\left(b_{2k+5}+b_{2k+3}\right)}=$$

$$=\frac{1}{b_1\cdot\left(q^{2n}-1\right)\cdot\left(q^4+q^2\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(b_{2k+5}+b_{2k+3}\right)}-\frac{1}{\sum_{k=1}^{n}\left(b_{2k+7}+b_{2k+5}\right)}\right].$$

$$\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)-\sum_{k=1}^{n}\left(b_{2k+2i+3}+b_{2k+2i+1}\right)=$$

$$=\sum_{k=1}^{n}\left(b_{2k+2i+5}-b_{2k+2i+1}\right)=\sum_{k=1}^{n}\left(b_1\cdot q^{2k+2i+4}-b_1\cdot q^{2k+2i}\right)=$$

$$=b_1\cdot q^{2i}\cdot\left(q^4-1\right)\left(q^2+q^4+q^6+...+q^{2n}\right)=$$

$$=b_1\cdot q^{2i}\left(q^2-1\right)\left(q^2+1\right)\cdot q^2\cdot\frac{q^{2n}-1}{q^2-1}=b_1\cdot q^{2i}\left(q^4+q^2\right)\left(q^{2n}-1\right)$$

$$+...+\frac{q^{2i}}{\sum_{k+1}^{n}\left(b_{2k+2i+3}+b_{2k+2i+1}\right)\cdot\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)}=$$

$$=\frac{1}{b_1\cdot\left(q^{2n}-1\right)\left(q^4+q^2\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(b_{2k+2i+3}+b_{2k+2i+1}\right)}-\frac{1}{\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)}\right].$$

$$S=\frac{1}{b_1\cdot\left(q^{2n}-1\right)\cdot\left(q^4+q^2\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(b_{2k+3}+b_{2k+1}\right)}-\frac{1}{\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)}\right]$$

$$\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)-\sum_{k=1}^{n}\left(b_{2k+3}+b_{2k+1}\right)=b_{2i+7}-b_5+b_{2i+5}-b_3+$$

$$+b_{2i+9}-b_7+b_{2i+7}-b_5+...+b_{2n+2i+5}-b_{2n+3}+b_{2n+2i+3}-b_{2n+1}=$$

$$=b_1\cdot q^{2i+6}-b_1\cdot q^4+b_1\cdot q^{2i+4}-b_1\cdot q^2+b_1\cdot q^{2i+8}-b_1\cdot q^6+b_1\cdot q^{2i+6}-b_1\cdot q^4+$$

$$+\ldots+b_1\cdot q^{2n+2i+4}-b_1\cdot q^{2n+2}+b_1\cdot q^{2n+2i+2}-b_1\cdot q^{2n}=$$

$$=b_1\cdot q^4\left(q^{2i+2}-1\right)+b_1\cdot q^2\cdot\left(q^{2i+2}-1\right)+$$
$$+b_1\cdot q^6\left(q^{2i+2}-1\right)+b_1\cdot q^4\left(q^{2i+2}-1\right)+$$

$$+\ldots+b_1q^{2n+2}\left(q^{2i+2}-1\right)+b_1\cdot q^{2n}\left(q^{2i+2}-1\right)=$$
$$=b_1\left(q^{2i+2}-1\right)\left(q^4+q^2\right)\cdot\frac{q^{2n}-1}{q^2-1}\Rightarrow$$

$$S=\frac{q^{2i+2}-1}{q^2-1}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(b_{2k+3}+b_{2k+1}\right)}-\frac{1}{\sum_{k=1}^{n}\left(b_{2k+2i+5}+b_{2k+2i+3}\right)}\right],\ \forall i\in N,\forall n\in N^*$$

2. Fie $(b_n)_{n\in N^*}\in R^*$ un şir de numere reale în progresie geometrică de raţie $q\neq 1$, atunci demonstraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n}b_{5k+3}\cdot\sum_{k=1}^{n}b_{5k+8}}+\frac{q^5}{\sum_{k=1}^{n}b_{5k+8}\cdot\sum_{k=1}^{n}b_{5k+13}}+\ldots+\frac{q^{5i}}{\sum_{k=1}^{n}b_{5k+5i+3}\cdot\sum_{k=1}^{n}b_{5k+5i+8}}=$$

$$=\frac{q^{5i+5}-1}{q^5-1}\cdot\left[\frac{1}{\sum_{k=1}^{n}b_{5k+3}}-\frac{1}{\sum_{k=1}^{n}b_{5k+5i+8}}\right],\ \forall n\in N^*,\ \forall i\in N.$$

Rezolvare:

$$\sum_{k=1}^{n}b_{5k+8}-\sum_{k=1}^{n}b_{5k+3}=b_{5n+8}-b_8=b_1\cdot q^{5n+7}-b_1\cdot q^7=b_1\cdot q^7\cdot\left(q^{5n}-1\right)$$

$$\frac{1}{\sum_{k=1}^{n}b_{5k+3}\cdot\sum_{k=1}^{n}b_{5k+8}}=\frac{1}{b_1\cdot q^7\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_{5k+3}}-\frac{1}{\sum_{k=1}^{n}b_{5k+8}}\right).$$

$$\sum_{k=1}^{n}b_{5k+13}-\sum_{k=1}^{n}b_{5k+8}=b_{5n+13}-b_{13}=b_1\cdot q^{5n+12}-b_1\cdot q^{12}=b_1\cdot q^{12}\cdot\left(q^{5n}-1\right)$$

$$\frac{q^5}{\sum_{k=1}^{n}b_{5k+8}\cdot\sum_{k=1}^{n}b_{5k+13}}=\frac{q^5}{b_1\cdot q^{12}\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_{5k+8}}-\frac{1}{\sum_{k=1}^{n}b_{5k+13}}\right).$$

$$\frac{q^5}{\sum_{k=1}^{n} b_{5k+8}\cdot\sum_{k=1}^{n} b_{5k+13}}=\frac{1}{b_1\cdot q^7\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n} b_{5k+8}}-\frac{1}{\sum_{k=1}^{n} b_{5k+13}}\right).$$

$$\sum_{k=1}^{n} b_{5k+5i+8}-\sum_{k=1}^{n} b_{5k+5i+3}=b_{5n+5i+8}-b_{5i+8}=b_1\cdot q^{5n+5i+7}-b_1\cdot q^{5i+7}=b_1\cdot q^{5i+7}\cdot\left(q^{5n}-1\right)$$

$$\frac{q^{5i}}{\sum_{k=1}^{n} b_{5k+5i+3}\cdot\sum_{k=1}^{n} b_{5k+5i+8}}=\frac{q^{5i}}{b_1\cdot q^{5i+7}\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n} b_{5k+5i+3}}-\frac{1}{\sum_{k=1}^{n} b_{5k+5i+8}}\right).$$

$$\frac{q^{5i}}{\sum_{k=1}^{n} b_{5k+5i+3}\cdot\sum_{k=1}^{n} b_{5k+5i+8}}=\frac{1}{b_1\cdot q^{7}\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n} b_{5k+5i+3}}-\frac{1}{\sum_{k=1}^{n} b_{5k+5i+8}}\right).$$

$$S=\frac{1}{b_1\cdot q^{7}\cdot\left(q^{5n}-1\right)}\cdot\left(\frac{1}{\sum_{k=1}^{n} b_{5k+3}}-\frac{1}{\sum_{k=1}^{n} b_{5k+5i+8}}\right)$$

$$\sum_{k=1}^{n} b_{5k+5i+8}-\sum_{k=1}^{n} b_{5k+3}=b_{5i+13}-b_8+b_{5i+18}-b_{13}+\ldots+b_{5n+5i+8}-b_{5n+3}=$$
$$=b_1\cdot q^{5i+12}-b_1\cdot q^7+b_1\cdot q^{5i+17}-b_1\cdot q^{12}+\ldots+b_1\cdot q^{5n+5i+7}-b_1\cdot q^{5n+2}=$$
$$=b_1\cdot q^7\cdot\left(q^{5i+5}-1\right)+b_1\cdot q^{12}\cdot\left(q^{5i+5}-1\right)+\ldots+b_1\cdot q^{5n+2}\cdot\left(q^{5i+5}-1\right)=$$
$$=b_1\cdot q^7\cdot\left(q^{5i+5}-1\right)\left(1+q^5+q^{10}+\ldots+q^{5n-5}\right)=$$
$$=b_1\cdot q^7\cdot\left(q^{5i+5}-1\right)\cdot\frac{q^{5n}-1}{q^5-1}$$

$$=\frac{1}{b_1\cdot q^{7}\cdot\left(q^{5n}-1\right)}\cdot\frac{b_1\cdot q^7\cdot\left(q^{5i+5}-1\right)\left(q^{5n}-1\right)}{q^5-1}\cdot\left(\frac{1}{\sum_{k=1}^{n} b_{5k+3}}-\frac{1}{\sum_{k=1}^{n} b_{5k+5i+8}}\right)$$

$$\frac{1}{\sum_{k=1}^{n} b_{5k+3}\cdot\sum_{k=1}^{n} b_{5k+8}}+\frac{q^5}{\sum_{k=1}^{n} b_{5k+8}\cdot\sum_{k=1}^{n} b_{5k+13}}+\ldots+\frac{q^{5i}}{\sum_{k=1}^{n} b_{5k+5i+3}\cdot\sum_{k=1}^{n} b_{5k+5i+8}}=$$

$$=\frac{q^{5i+5}-1}{q^5-1}\cdot\left[\frac{1}{\sum_{k=1}^{n} b_{5k+3}}-\frac{1}{\sum_{k=1}^{n} b_{5k+5i+8}}\right],\ \forall n\in N^*,\ \forall i\in N.$$

3. Fie $(b_n)_{n\in N^*}\in R^*$ un şir de numere reale în progresie geometrică de raţie $q\neq 1$, atunci demonstraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n}b_k\cdot\sum_{k=1}^{n}b_{k+1}}+\frac{q}{\sum_{k=1}^{n}b_{k+1}\cdot\sum_{k=1}^{n}b_{k+2}}+\frac{q^2}{\sum_{k=1}^{n}b_{k+2}\cdot\sum_{k=1}^{n}b_{k+3}}+\ldots+\frac{q^i}{\sum_{k=1}^{n}b_{k+i}\cdot\sum_{k=1}^{n}b_{k+i+1}}=$$

$$=\frac{q^{i+1}-1}{q-1}\cdot\frac{1}{\sum_{k=1}^{n}b_k\cdot\sum_{k=1}^{n}b_{k+i+1}},\ \forall n\in N^*,\forall i\in N.$$

Rezolvare:

$$\sum_{k=1}^{n}b_{k+1}-\sum_{k=1}^{n}b_k=b_{n+1}-b_1=b_1\cdot q^n-b_1=b_1\cdot(q^n-1)$$

$$\frac{1}{\sum_{k=1}^{n}b_k\cdot\sum_{k=1}^{n}b_{k+1}}=\frac{1}{b_1\cdot(q^n-1)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_k}-\frac{1}{\sum_{k=1}^{n}b_{k+1}}\right)$$

$$\sum_{k=1}^{n}b_{k+2}-\sum_{k=1}^{n}b_{k+1}=b_{n+2}-b_2=b_1\cdot q^{n+1}-b_1\cdot q=b_1\cdot q\cdot(q^n-1)$$

$$\frac{q}{\sum_{k=1}^{n}b_{k+1}\cdot\sum_{k=1}^{n}b_{k+2}}=\frac{1}{b_1\cdot(q^n-1)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_{k+1}}-\frac{1}{\sum_{k=1}^{n}b_{k+2}}\right)$$

$$\sum_{k=1}^{n}b_{k+3}-\sum_{k=1}^{n}b_{k+2}=b_{n+3}-b_3=b_1\cdot q^{n+2}-b_1\cdot q^2=b_1\cdot q^2\cdot(q^n-1)$$

$$\frac{q^2}{\sum_{k=1}^{n}b_{k+2}\cdot\sum_{k=1}^{n}b_{k+3}}=\frac{1}{b_1\cdot(q^n-1)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_{k+2}}-\frac{1}{\sum_{k=1}^{n}b_{k+3}}\right)$$

$$\sum_{k=1}^{n}b_{k+i+1}-\sum_{k=1}^{n}b_{k+i}=b_{n+i+1}-b_{i+1}=b_1\cdot q^{n+i}-b_1\cdot q^i=b_1\cdot q^i\cdot(q^n-1)$$

$$\frac{q^i}{\sum_{k=1}^{n}b_{k+i}\cdot\sum_{k=1}^{n}b_{k+i+1}}=\frac{1}{b_1\cdot(q^n-1)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_{k+i}}-\frac{1}{\sum_{k=1}^{n}b_{k+i+1}}\right)$$

$$S=\frac{1}{b_1\cdot(q^n-1)}\cdot\left(\frac{1}{\sum_{k=1}^{n}b_k}-\frac{1}{\sum_{k=1}^{n}b_{k+i+1}}\right)$$

$$\sum_{k=1}^{n} b_{k+i+1} - \sum_{k=1}^{n} b_k = b_{i+2} - b_1 + b_{i+3} - b_2 + \ldots + b_{n+i+1} - b_n =$$
$$= b_1 \cdot \left(q^{i+1} - 1\right) + b_1 \cdot q \cdot \left(q^{i+1} - 1\right) + \ldots + b_1 \cdot q^{n-1} \cdot \left(q^{i+1} - 1\right) =$$
$$= b_1 \cdot \left(q^{i+1} - 1\right)\left(1 + q + q^2 + \ldots + q^{n-1}\right) = b_1 \cdot \left(q^{i+1} - 1\right) \cdot \frac{q^n - 1}{q - 1}$$

$$S = \frac{1}{b_1 \cdot \left(q^n - 1\right)} \cdot \frac{b_1 \cdot \left(q^{i+1} - 1\right)\left(q^n - 1\right)}{q - 1} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_k} \cdot \frac{1}{\sum_{k=1}^{n} b_{k+i+1}} \right) \Rightarrow$$

$$\frac{1}{\sum_{k=1}^{n} b_k \cdot \sum_{k=1}^{n} b_{k+1}} + \frac{q}{\sum_{k=1}^{n} b_{k+1} \cdot \sum_{k=1}^{n} b_{k+2}} + \frac{q^2}{\sum_{k=1}^{n} b_{k+2} \cdot \sum_{k=1}^{n} b_{k+3}} + \ldots + \frac{q^i}{\sum_{k=1}^{n} b_{k+i} \cdot \sum_{k=1}^{n} b_{k+i+1}} =$$
$$= \frac{q^{i+1} - 1}{q - 1} \cdot \frac{1}{\sum_{k=1}^{n} b_k \cdot \sum_{k=1}^{n} b_{k+i+1}}, \forall n \in N^*, \forall i \in N.$$

4. Fie $(b_n)_{n\geq 1} \in R^*$ un şir de numere reale în progresie geometrică de raţie $q \neq 1$ şi $(a_m)_{m\geq 1} \in N^*$ un şir de numere naturale în progresie aritmetică de raţie $r \in N^*$, atunci demostraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n} b_{a_k} \cdot \sum_{k=1}^{n} b_{a_{k+1}}} + \frac{q^r}{\sum_{k=1}^{n} b_{a_{k+1}} \cdot \sum_{k=1}^{n} b_{a_{k+2}}} +$$
$$+ \frac{q^{ir}}{\sum_{k=1}^{n} b_{a_{k+i}} \cdot \sum_{k=1}^{n} b_{a_{k+i+1}}} = \frac{q^{(i+1)r} - 1}{q^r - 1} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_k}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+i+1}}} \right), \forall i \in N, \forall n \in N^*$$

Rezolvare:

$$\sum_{k=1}^{n} b_{a_{k+1}} - \sum_{k=1}^{n} b_{a_k} = b_{a_{n+1}} - b_{a_1} =$$
$$= b_1 \cdot q^{a_{n+1}-1} - b_1 \cdot q^{a_1 - 1} = b_1 \cdot q^{a_1 + nr - 1} - b_1 \cdot q^{a_1 - 1} =$$
$$= b_1 \cdot q^{a_1 - 1}\left(q^{nr} - 1\right)$$

$$\frac{1}{\sum_{k=1}^{n} b_{a_k} \cdot \sum_{k=1}^{n} b_{a_{k+1}}} = \frac{1}{b_1 \cdot q^{a_1 -1}\left(q^{nr}-1\right)} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_k}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+1}}} \right)$$

$$\sum_{k=1}^{n} b_{a_{k+2}} - \sum_{k=1}^{n} b_{a_{k+1}} = b_{a_{n+2}} - b_{a_2} =$$

$$= b_1 \cdot q^{a_{n+2}-1} - b_1 \cdot q^{a_2 -1} = b_1 \cdot q^{a_1 + (n+1)r - 1} - b_1 \cdot q^{a_1 + r - 1} =$$

$$= b_1 \cdot q^{a_1 - 1} \cdot q^{r}\left(q^{nr} - 1\right)$$

$$\frac{q^r}{\sum_{k=1}^{n} b_{a_{k+1}} \cdot \sum_{k=1}^{n} b_{a_{k+2}}} = \frac{1}{b_1 \cdot q^{a_1 -1} \cdot \left(q^{nr}-1\right)} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_{k+2}}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+1}}} \right)$$

$$\sum_{k=1}^{n} b_{a_{k+i+1}} - \sum_{k=1}^{n} b_{a_{k+i}} = b_{a_{n+k+i}} - b_{a_{i+1}} =$$

$$= b_1 \cdot q^{a_{n+i+1}-1} - b_1 \cdot q^{a_{i+1} -1} = b_1 \cdot q^{a_1 + (n+i)r - 1} - b_1 \cdot q^{a_1 + ir - 1} =$$

$$= b_1 \cdot q^{a_1 - 1} \cdot q^{ir}\left(q^{nr} - 1\right)$$

$$\frac{q^{ir}}{\sum_{k=1}^{n} b_{a_{k+i}} \cdot \sum_{k=1}^{n} b_{a_{k+i+1}}} = \frac{1}{b_1 \cdot q^{a_1 -1} \cdot \left(q^{nr}-1\right)} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_{k+i}}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+i+1}}} \right)$$

$$S = \frac{1}{b_1 \cdot q^{a_1 -1}\left(q^{nr}-1\right)} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_k}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+i+1}}} \right)$$

$$\sum_{k=1}^{n} b_{a_{k+i+1}} - \sum_{k=1}^{n} b_{a_k} = b_{a_{i+2}} - b_{a_1} + b_{a_{i+3}} - b_{a_2} + \ldots + b_{a_{n+i+1}} - b_{a_n} =$$

$$= b_1 \cdot q^{a_{i+2}-1} - b_1 \cdot q^{a_1-1} + b_1 \cdot q^{a_{i+3}-1} - b_1 \cdot q^{a_2-1} + \ldots + b_1 \cdot q^{a_{n+i+1}-1} - b_1 \cdot q^{a_n-1} =$$

$$= b_1 \cdot q^{a_1+(i+1)r-1} - b_1 \cdot q^{a_1-1} + \ldots + b_1 \cdot q^{a_1+(n+i)r-1} - b_1 \cdot q^{a_1+(n-1)r-1} =$$

$$= b_1 \cdot q^{a_1-1}\left[q^{(i+1)r} - 1\right] \cdot \left[1 + q^r + \ldots + q^{(n-1)r}\right] =$$

$$= b_1 \cdot q^{a_1-1}\left[q^{(i+1)r} - 1\right] \cdot \frac{q^{nr}-1}{q^r-1}$$

$$\Rightarrow \frac{1}{\sum_{k=1}^{n} b_{a_k} \cdot \sum_{k=1}^{n} b_{a_{k+1}}} + \frac{q^r}{\sum_{k=1}^{n} b_{a_{k+1}} \cdot \sum_{k=1}^{n} b_{a_{k+2}}} +$$

$$+ \frac{q^{ir}}{\sum_{k=1}^{n} b_{a_{k+i}} \cdot \sum_{k=1}^{n} b_{a_{k+i+1}}} = \frac{q^{(i+1)r}-1}{q^r-1} \cdot \left(\frac{1}{\sum_{k=1}^{n} b_{a_k}} - \frac{1}{\sum_{k=1}^{n} b_{a_{k+i+1}}} \right), \forall i \in N, \forall n \in N^*$$

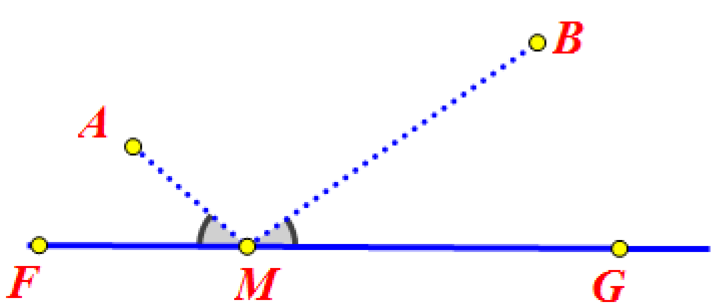
5. PROBLEMA BILIARDULUI

Prof. Andrei Dobre

Să considerăm două bile aşezate pe o masa de billiard, ca în figura urmatoare. În ce direcţie trebuie lovită bila asezata in punctual *A* astfel incat dupa ciocnirea cu o latură a mesei să lovească bila aşezată în punctul *B*?

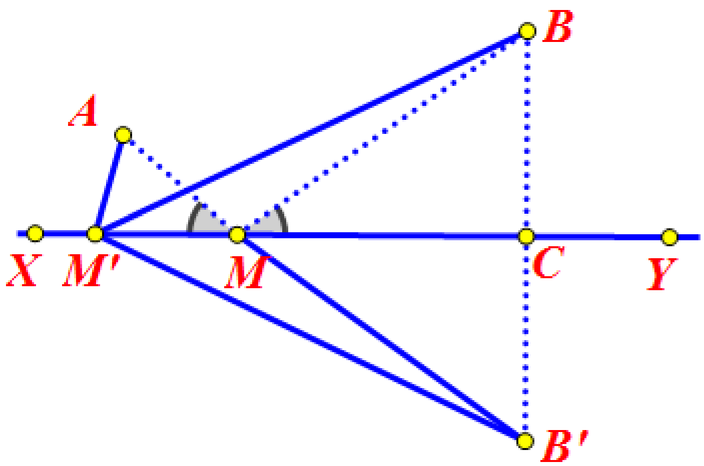
Ştim de la fizică că,în cazul unei ciocniri elastic, unghiurile (ascutite) facute de AM şi BM cu respective latură sunt congruente. Astfel, am putea reformula problema noastră in limbaj matematic în felul urmator:

1.Fie *A* şi *B* doua puncte situate de aceeaşi parte a unei drepte *FG* (ca în figura de mai jos). Să se determine poziţia unui punct *M* pe dreaptă astfel încât unghiurile $\angle AMF$si $\angle BMG$ sa fie congruente.



În locul acestei probleme, vom formula alta, echivalentă (după cum vom vedea) cu ea:

2.Fie *A* si *B* două puncte situate de aceeaşi parte a unei drepte *XY*. Sa se determine poziţia unui punct *M* pe dreaptă astfel încât suma *AM+MB* să fie minima.

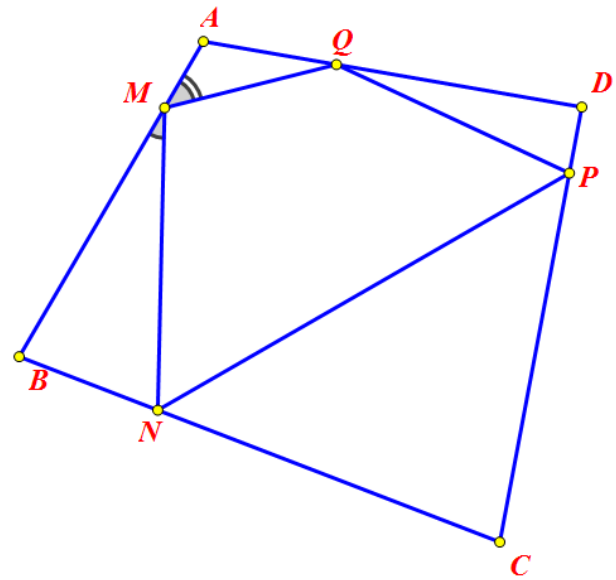


Soluţie: Fie *B'* simetricul punctului *B* faţă de dreapta şi *M'* un punct arbitrar pe acesta. Din simetrie, *BM'= B'M'*, prin urmare rezultă că*AM'+M'B=AM'+M'B'*. Ultima sumă este minimă atunci când punctele *A, M'* si *B'* sunt coliniare, aşadar punctul căutat este punctul *M*, intersecţia dintre dreaptă şi segmentul*AB'*.

Dacă dreapta ar reprezenta suprafaţa unei oglinzi, atunci o raza de lumina care pleacă din punctul *A* în direcţia punctului *M* se va reflecta de oglinda şi, respectând legile reflexiei, va ajunge în punctul *B*. Astfel, înţelegem de ce se spune ca "lumina merge pe drumul cel mai scurt".

Să examinăm câteva probleme în care sunt folosite ideile precedente.

3.Fie *ABCD* un patrulater convex. Să se arate că dacă există punctele *M, N, P, Q* situate, respective, în interiorul laturilor *AB, BC, CD, DA* astfel încât perimetrul patrulaterului *MNPQ* să fie minim, atunci *ABCD* este un patrulater inscriptibil.



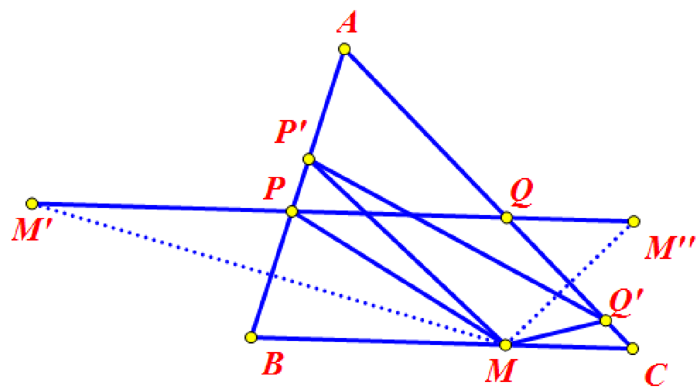
Soluţie: Mai întâi, se pune întrebarea: nu întotdeauna există punctele *M, N, P, Q* având proprietatea ceruta, indiferent dacă*ABCD* este inscriptibil sau nu? Da , există întotdeauna , dar nu neapărat în interiorul laturilor respective .Dacă luăm în considerație doar patrulaterele MNPQ pentru care punctele sunt în interiorul laturilor lui ABCD , este posibil să nu existe un patrulater având perimetrul minim (în mod similar , nu există un număr minim în intervalul deschis (0,1)!) .

Revenind la problemă , să observăm că dacă presupunem că perimetrul patrulaterului MNPQ este minim , atunci $\angle AMQ \equiv \angle BMN$, $\angle BNM \equiv \angle CNP$, $\angle CPN \equiv \angle DPQ$ și , în fine , $\angle DQP \equiv \angle AQM$.

Într-adevăr , dacă , de exemplu , $\angle AMQ \equiv \angle BMN$, putem deplasa punctul M pe latura AB astfel încât suma QM+MN să se micșoreze (demonstrați această afirmație !) , și , deci , perimetrul patrulaterului MNPQ să fie mai mic .

Un calcul simplu al măsurilor unghiurilor conduce la egalitatea m($\angle$A) +m($\angle$C)=m($\angle$B)+m($\angle$D) , ceea ce reprezintă condiția necesară și suficientă pentru ca patrulaterul ABCD să fie inscriptibil.

4. Se consideră un triunghi ascuțitunghic ABC și M un punct situat în interiorul laturii BC . Să se determine pozițiile punctelor P și Q pe laturile AB și AC astfel încât perimetrul triunghiului MPQ să fie minim .



Soluție Fie M' și M'' simetricele punctului M față de laturile AB și , respectiv, AC . Punctele P și Q căutate sunt punctele de intersecție sintre dreapta M'M'' și laturile AB și AC. Să observăm că , într-adevăr , luând alte puncte , P 'și Q' pe AB și AC , din cauza simetriei , perimetrul triunghiului MP'Q', MP'+P'Q'+Q'M este egal cu M'P'+P'Q'+Q'M'' , iar lungimea liniei frânte M'-P'-Q'-M' este minima când punctele M',P,'Q',M'' sunt coliniare , adică atunci când P'=P și Q'=Q.

Observație. Se poate demonstra că triunghiul cu perimetru minim ale cărui vârfuri sunt situate pe laturile triunghiului ABC este triunghiul ortic (triunghiul ale cărui vârfuri sunt picioarele înălțimilor

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6.Asupra unor probleme cu matrici

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1) Calculaţi determinantul $D=\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_1 & x_2 & x_3 \end{vmatrix}$,

unde x_1, x_2, x_3, x_4 sunt soluţiile ecuaţiei $x^4+ax^2+bx+c=0$.

Rezolvare: Adunând toate coloanele la prima se obţine:

$D=(x_1+x_2+x_3+x_4)\begin{vmatrix} 1 & x_2 & x_3 & x_4 \\ 1 & x_3 & x_4 & x_1 \\ 1 & x_4 & x_1 & x_2 \\ 1 & x_1 & x_2 & x_3 \end{vmatrix}$ şi din relaţiile lui Viète rezultă $x_1+x_2+x_3+x_4=0$. Deci $D=0$.

2) Fie $M=\left\{A=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, unde\ x,y\in\mathbb{R}\right\}$ şi funcţia $f:\mathbb{C}\to M$, $f(x+iy)=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$.

Arătaţi că:

a) f este bijectivă;

b) $f(z+z')=f(z)+f(z')$ şi $f(zz')=f(z)\cdot f(z')$, oricare ar fi $z,z'\in\mathbb{C}$.

Rezolvare:

a) Se verifică faptul că funcţia $g:M\to\mathbb{C}$ definită prin $g\left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right)=x+yi$ este inversa funcţiei f;

b) Fie $z=x+iy$ şi $z'=x'+iy'$.

$f(z+z')=f((x+x')+i(y+y'))=\begin{pmatrix} x+x' & y+y' \\ -(y+y') & x+x' \end{pmatrix}$, iar

$f(z)+f(z')=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}+\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix}=\begin{pmatrix} x+x' & y+y' \\ -(y+y') & x+x' \end{pmatrix}$, de unde rezultă că

$f(z+z')=f(z)+f(z')$, $\forall\ z,z'\in\mathbb{C}$;

$f(z\cdot z')=f((xx'-yy')+i(xy'+x'y))=\begin{pmatrix} xx'-yy' & xy'+x'y \\ -(xy'+x'y) & xx'-yy' \end{pmatrix}$, iar

$f(z)\cdot f(z')=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\cdot\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix}=\begin{pmatrix} xx'-yy' & xy'+x'y \\ -(xy'+x'y) & xx'-yy' \end{pmatrix}$, de unde rezultă că $f(zz')=f(z)\cdot f(z')$, $\forall\ z,z'\in\mathbb{C}$.

3) Fie matricea $A=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ astfel încât $0\le x^2+y^2<1$.

a) Arătați că matricea A^n este de forma $\begin{pmatrix} x_n & y_n \\ -y_n & x_n \end{pmatrix}$;

b) Demonstrați că șirurile x_n și y_n sunt convergente și au limita zero.

Rezolvare:

a) Din exercițiul 2) rezultă că A^n este de forma $\begin{pmatrix} x_n & y_n \\ -y_n & x_n \end{pmatrix}$, cu $x_n+iy_n=(x+iy)^n$;

b) $0\le x^2+y^2<1$ este echivalentă cu $|x+iy|<1$. Atunci șirul de numere complexe $z_n=(x+iy)^n$ tinde la zero, ceea ce înseamnă că $x_n\to 0$ și $y_n\to 0$.

Pentru a nu folosi convergența șirurilor de numere complexe se utilizează scrierea trigonometrică: $x+iy=\rho(\cos\theta+i\sin\theta)$. Din ipoteză rezultă că $\rho<1$, iar din a) și formula lui Moivre se obține: $x_n=\rho^n\cos n\theta$, $y_n=\rho^n\sin n\theta$. Astfel că $|x_n|=\rho^n|\cos n\theta|\le\rho^n$ și $|y_n|=\rho^n|\sin n\theta|$, de unde $x_n\to 0$, $y_n\to 0$, deoarece $\lim\limits_{n\to\infty}\rho^n=0$.

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