OLIMPIADA NAȚIONALĂ DE MATEMATICĂ

ETAPA LOCALĂ - BRĂILA, 22 februarie 2015

CLASA A IX-A - Soluții

1. Fie triunghiul ABC înscris într-un cerc de centru O. Fie P şi Q simetricele ortocentrului şi a vârfului A față de mijlocul lui (BC). Demonstrați că $\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OP}$

Gazeta Matematică

Soluție. Fie M mijlocul lui (BC) şi H ortocentrul triunghiului ABC, atunci avem

$$\left.\begin{array}{l} 2\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{OH} \\ 2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OQ} \end{array}\right\} \Rightarrow \overrightarrow{OA} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OH} \Leftrightarrow \overrightarrow{OA} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \Rightarrow \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OB} + \overrightarrow{OC}$$

2. Fie $n \in Z^*$. Aflați $x \in Z$ pentru care $\dfrac{x^{5n} + 2x^{2n} - x^n + 1}{x^{5n} + x^n - 1} \in Z.$

Carmen şi Viorel Botea, Brăila

Soluție. Notăm $x^n = y \Rightarrow$

$$\Rightarrow \frac{y^5 + 2y^2 - y + 1}{y^5 + y - 1} = \frac{\left(y^5 + y^2\right) + \left(y^2 - y + 1\right)}{\left(y^5 + y^2\right) - \left(y^2 - y + 1\right)} = \frac{y^2(y+1)\left(y^2 - y + 1\right) + \left(y^2 - y + 1\right)}{y^2(y+1)\left(y^2 - y + 1\right) - \left(y^2 - y + 1\right)} = \frac{y^3 + y^2 + 1}{y^3 + y^2 - 1} \in Z \Rightarrow$$

$$\Rightarrow \begin{cases} y^3 + y^2 - 1 \,\middle|\, y^3 + y^2 + 1 \\ y^3 + y^2 - 1 \,\middle|\, y^3 + y^2 - 1 \end{cases} \Rightarrow \underbrace{y^3 + y^2}_{par} - 1 \,\Big|\, 2 \Rightarrow y^3 + y^2 - 1 = \pm 1 \Rightarrow$$

$\Rightarrow$ i) $y^3 + y^2 - 1 = -1 \Rightarrow y = 0, y = -1 \Rightarrow x = 0, x = -1, n$ impar

ii) $y^3 + y^2 - 1 = 1 \Rightarrow y = 1 \Rightarrow x = \pm 1, n$ par şi $x = 1, n$ impar,

deci $x \in \{+1, 0, 1\}, (\forall n) \in Z$.

3. Fie şirul $(x_n)_{n \geq 1}, x_1 = \dfrac{1}{2014}, x_{n+1} = x_n\left(1 + x_1 + x_1^2 + \ldots + x_1^n\right)$, oricare ar fi $n \geq 1$. Notăm cu $S = \dfrac{x_1}{x_2} + \dfrac{x_2}{x_3} + \ldots + \dfrac{x_{2014}}{x_{2015}}$. Aflați $[S]$.

Carmen şi Viorel Botea, Brăila

Soluție. $\dfrac{x_n}{x_{n+1}}=\dfrac{1}{1+x_1+x_1^2+\ldots+x_1^n},(\forall)n\geq 1$.

$$S=\frac{1}{1+\frac{1}{2014}}+\frac{1}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2}+\ldots+\frac{1}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+\ldots+\left(\frac{1}{2014}\right)^{2014}}<2014\frac{1}{1+\frac{1}{2014}}=$$

$$=\frac{2014^2}{2015}<2014\Rightarrow S<2014$$

$$S>\frac{2014}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+\ldots+\left(\frac{1}{2014}\right)^{2014}};1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+\ldots+\left(\frac{1}{2014}\right)^{2014}=\frac{1-\left(\frac{1}{2014}\right)^{2015}}{1-\frac{1}{2014}}<$$

$$<\frac{2014}{2013}\Rightarrow S>\frac{2014\cdot 2013}{2014}=2013\Rightarrow[S]=2013$$

4. Fie triunghiul oarecare ABC. Considerăm dreptele $d_1\parallel AB,\ d_2\parallel AC,\ d_3\parallel BC$ şi notăm $d_1\cap(AC)=\{I\},\ d_1\cap(BC)=\{F\},\ d_2\cap(AB)=\{D\},\ d_2\cap(BC)=\{G\},\ d_3\cap(AB)=\{E\}$, $d_3\cap(AC)=\{H\}$, $d_1\cap d_2=\{A'\},\ d_1\cap d_3=\{B'\},\ d_2\cap d_3=\{C'\}$. Presupunem că $\text{Aria}(AEH)=\text{Aria}(CIF)=\text{Aria}(BDG)=\text{Aria}(ADGC)=\text{Aria}(BEHC)=\text{Aria}(AIFB)$.

Aflați $\dfrac{Aria(A'B'C')}{Aria(ABC)}$.

Elev Paul Crestez, Brăila

Soluție. Deoarece

$$EH\parallel BC\Rightarrow \Delta AEH asemenea \Delta ABC\Rightarrow\left(\frac{AE}{AB}\right)^2=\frac{Aria(\Delta AEH)}{Aria(\Delta ABC)}=\frac{Aria(\Delta AEH)}{Aria(\Delta AEH)+Aria(\Delta EBCH)}=$$

$$=\frac{Aria(\Delta AEH)}{2Aria(\Delta AEH)}=\frac{1}{2}\Rightarrow\frac{AE}{AB}=\frac{1}{\sqrt{2}}\Rightarrow AE=\frac{AB}{\sqrt{2}}\Rightarrow EB=AB-AE=AB-\frac{AB}{\sqrt{2}}=AB\frac{\sqrt{2}-1}{\sqrt{2}}.$$

$EH\parallel BC$ şi $AB\parallel IF\Rightarrow EB'FB$ paralelogram $\Rightarrow B'F=EB=AB\dfrac{\sqrt{2}-1}{\sqrt{2}},(1)$. Deoarece $DG\parallel AC\Rightarrow$

$$\Rightarrow\Delta BDG asemenea \Delta BAC\Rightarrow AD=AB\frac{\sqrt{2}-1}{\sqrt{2}}.$$

$IF\parallel AB$ şi $DG\parallel AC\Rightarrow AIA'D$ paralelogram $\Rightarrow A'I=AD=AB\dfrac{\sqrt{2}-1}{\sqrt{2}},(2)$.

$$IF \parallel AB \Rightarrow \Delta CIF asemenea \Delta CAB \Rightarrow \left(\frac{IF}{AB}\right)^2 = \frac{Aria\left(\Delta CIF\right)}{Aria\left(\Delta CAB\right)} = \frac{Aria\left(\Delta CIF\right)}{Aria\left(\Delta CIF\right) + Aria\left(IFBA\right)} = \frac{Aria\left(\Delta CIF\right)}{2Aria\left(\Delta CIF\right)} =$$

$= \frac{1}{2} \Rightarrow \frac{IF}{AB} = \frac{1}{\sqrt{2}} \Rightarrow IF = \frac{AB}{\sqrt{2}}, (3)$. Din $(1), (2)(3)$ avem $A^/ B^/ = IF - IA^/ - IB^/ = \frac{AB}{\sqrt{2}} - AB\frac{\sqrt{2}-1}{\sqrt{2}} -$

$-AB\frac{\sqrt{2}-1}{\sqrt{2}} = AB\frac{3-2\sqrt{2}}{\sqrt{2}}$.

$$\left.\begin{array}{l} A^/ B^/ \parallel AB \\ A^/ C^/ \parallel AC \\ B^/ C^/ \parallel BC \end{array}\right\} \Rightarrow \Delta ABC asemenea \Delta A^/ B^/ C^/ \Rightarrow \frac{Aria\left(\Delta A^/ B^/ C^/\right)}{Aria\left(\Delta ABC\right)} = \left(\frac{A^/ B^/}{AB}\right)^2 \Rightarrow \frac{Aria\left(\Delta A^/ B^/ C^/\right)}{Aria\left(\Delta ABC\right)} = \left(\frac{3-2\sqrt{2}}{\sqrt{2}}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{Aria\left(\Delta A^/ B^/ C^/\right)}{Aria\left(\Delta ABC\right)} = \frac{17-12\sqrt{2}}{2}.$$