OLIMPIADA NAȚIONALĂ DE MATEMATICĂ

ETAPA LOCALĂ - BRĂILA, 22 februarie 2015

CLASA A XII-A - Soluții

1.

Inmultim relatia data cu e^{-x}

Si obtinem $e^{-x}[f(x) - F(x)] = \dfrac{e^{x}}{e^{2x}+1}$

Aceasta relatie se rescrie $(e^{-x}[F(x)])' = \dfrac{e^{x}}{e^{2x}+1}$

$e^{-x}\left[F(x)\right] = \int \dfrac{e^{x}}{e^{2x}+1}\,dx$

$\int \dfrac{e^{x}}{e^{2x}+1}$ =arctg $e^{x} + c$ unde $c \in \mathbf{R}$

Deci $F(x) = e^{x}$ arctg $e^{x} + ce^{x}$ si prin derivarea relatiei obtinem $f(x)$

2.

1) „$*$" *se poate scrie* : $x * y = \dfrac{k + \frac{1}{k}\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}{1 + \left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}$

Se verifica axiomele grupului

$-$*element neutru* : $e = \dfrac{k^2+1}{2k} \in G$

$-$*element simetric* : $x' = k + \dfrac{1}{k} - x \in G$
$(\forall) x \in G$

2) $\underbrace{x * x * \ldots * x}_{\text{„}n\text{" ori}} = x^{(n)} = \dfrac{k + (-1)^n \cdot \frac{1}{k} \cdot \left(\frac{x-k}{x-\frac{1}{k}}\right)^n}{1 + (-1)^n \cdot \left(\frac{x-k}{x-\frac{1}{k}}\right)^n}$ $\quad n \geq 2, n \in N$

Se demonstreaza prin inductie completa

2) *Partile stabile finite ale lui G*

$(\exists) m, n \in N^*, m \neq n \quad a.t. \quad x^{(n)} = x^{(m)}$

$Fie\ t = \dfrac{x - k}{x - \frac{1}{k}}, <=> (-t)^m = (-t)^n$

$Fie\ m > n \quad (-t)^n * [(-t)^{m-n} - 1] = 0$

$<=> a) \quad (-t)^n = 0 \quad sau \quad b) \quad (-t)^{m-n} = 1$

$a) \quad (-t)^n = 0 <=> -t = 0 \qquad \dfrac{x-k}{x - \frac{1}{k}} = 0 <=> x = k \quad (fals)$

$$x \in \left(\frac{1}{k}, k\right)$$

$b) \quad (-t)^{m-n} = 1$

$1)\ m - n = impar => -t = 1 <=> t = -1$

$\dfrac{x-k}{x-\frac{1}{k}} = -1 => x - k = -x + \dfrac{1}{k}$

$2x = k + \dfrac{1}{k} \quad => \quad x = \dfrac{k^2+1}{2k} = e \in \left(\dfrac{1}{k}, k\right)$

$2) m - n = par => -t = \pm 1 \quad =>$

$t = 1 \quad sau \quad t = -1$

$\dfrac{x-k}{x-\frac{1}{k}} = 1 => -k = -\dfrac{1}{k}, k = \dfrac{1}{k}\ fals$

$\dfrac{x-k}{x-\frac{1}{k}} = -1 => x = \dfrac{k^2+1}{2k} = e \in G$

Deci singura parte stabila finita a lui G fata de „" este* $E = \{e\} = \left\{\dfrac{k^2+1}{2k}\right\}$

E este un subgrup finit impropriu a lui G si este singurul.

$(E,*) subgrup \quad \subset \quad (G,*) grup$

$3) \quad f: (G,*) \ -> \ (G,*)$

f automorfism strict descrescator cautam un automorfism de forma:

$f(x) = m * x + n$

$1) \ f\ bijectiva\ pe \left(\dfrac{1}{k}, k\right) - G$

$(\forall) x, y \in G$

$2) \ f(x * y) = f(x) * f(y)$

X	$\frac{1}{k}$	$\frac{k^2+1}{2k}$	k

f(x)	k	$\searrow$	$\frac{k^2+1}{2k}$	$\searrow$	$\frac{1}{k}$

$$\lim_{\substack{x->\frac{1}{k}\\ x>\frac{1}{k}}}(m*x+n)=\frac{m}{k}+n=k=>m+k*n=k^2$$

$$\lim_{\substack{x->k\\ x<k}}(m*x+n)=m*k+n=\frac{1}{k},m*k^2+n*k=1$$

$$\begin{cases} m+k*n=k^2\\ m*k^n+k*n=1\end{cases}\quad => \quad \begin{cases} m=-1\\ n=\dfrac{k^2+1}{k}\end{cases}$$

$$f(x)=-x+\frac{k^2+1}{k}\ ,\qquad f:\left(\frac{1}{k},k\right)->\left(\frac{1}{k},k\right)$$

$$f \searrow pe \left(\frac{1}{k},k\right)=>f\ injectiva\ pe\ G$$

$$f\ cont\ si\ f\left[\left(\frac{1}{k},k\right)\right]=\left(\frac{1}{k},k\right)=>f\ surjectiva\ pe\ G$$

$$=>1)\ f\ bijectiva\ pe\ G$$

$$\quad 2)\ f(x*y)=f(x)*f(y),\quad (\forall)x,y\in G$$

$$f(x*y)=-(x*y)+\frac{k^2+1}{k}=$$

$$=-\left[\frac{k+\frac{1}{k}\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}{1+\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}\right]+k+\frac{1}{k}=$$

$$=\frac{\frac{1}{k}+k\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}{1+\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}=f(x*y)$$

$$f(x)*f(y)=\frac{k+\frac{1}{k}\left[\frac{f(x)-k}{f(x)-\frac{1}{k}}\right]\cdot\left[\frac{f(y)-k}{f(y)-\frac{1}{k}}\right]}{1+\left[\frac{f(x)-k}{f(x)-\frac{1}{k}}\right]\cdot\left[\frac{f(y)-k}{f(y)-\frac{1}{k}}\right]}=$$

$$=\frac{\frac{1}{k}+k\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}{1+\left(\frac{x-k}{x-\frac{1}{k}}\right)\cdot\left(\frac{y-k}{y-\frac{1}{k}}\right)}=f(x*y)$$

$=> f(x*y)=f(x)*f(y) \quad (\forall)x,y \in G$

$$\left.\begin{array}{l}-f \text{ bijectiva pe } G\\ -f \text{ endomorfism pe } (G,*)\end{array}\right\}=>$$

$=> f$ automorfism pe grupul $(G,*)$ strict descrescator

3. Fie $x\in G\setminus H \Rightarrow x^{-1}\in G\setminus H$ şi $x\neq x^{-1}$ deoarece grupul are ordin impar.Cum x şi x^{-1} comută, problema este rezolvată.

4. $f:\left(\frac{\pi}{24},\frac{\pi}{3}\right)\to R$, $f(x)=e^{tgx}+\sin^2 x$

$$f'(x)=\frac{e^{tgx}}{\cos^2 x}+\sin 2x$$

$$f'(x)-\frac{f(x)}{\cos^2 x}=\sin 2x-tg^2x$$

$$\int\frac{\sin 2x-tg^2x}{e^{tgx}+\sin^2 x}dx==\int\frac{f'(x)-\frac{f(x)}{\cos^2 x}}{f(x)}dx=\int\left(\frac{f'(x)}{f(x)}-\frac{1}{\cos^2 x}\right)dx=\ln\left(e^{tgx}+\sin^2 x\right)-tgx+C$$